Enhanced parameter estimation in adaptive control via online historical data

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Abstract: Parameter convergence is desirable for adaptive control as it enhances the overall stability and robustness properties of the closed-loop system. In existing online historical data (OHD)-driven parameter estimation schemes, all OHD are exploited to update parameter estimates such that exponential parameter convergence is ensured under a condition of sufficient excitation which is strictly weaker than the traditional persistent excitation (PE) condition. However, the exploitation of all OHD not only results in possible unbounded adaptation but also loses the flexibility of handling slow time-varying uncertainties. In this brief, a novel OHD-driven parameter estimation scheme that exploits only partial OHD is presented to improve parameter convergence and is incorporated with direct adaptive control to construct a composite learning control strategy. The proposed approach guarantees exponential parameter convergence under a condition of interval excitation which is also strictly weaker than the PE condition while eliminating the drawbacks of existing OHD-driven parameter estimation schemes. Numerical results have verified the effectiveness and superiority of the proposed approach.

1 Introduction

Parameter estimation is regarded as the foundation of adaptive control. Conventionally, adaptive control is focused on the convergence of tracking errors, where the parameter estimation algorithms therein do not guarantee convergence of parameter estimation errors at the absence of a condition termed persistent excitation (PE) [1]. Parameter convergence is desirable as it enhances the overall stability and robustness properties of adaptive control systems [2]. Although the verifiability of the PE condition has been well studied for parameter estimation in adaptive control of both linear systems [3–5] and non-linear systems [2, 6–8], it is difficult to be satisfied in practice as the excitation has to be persistent for all time [9]. Even if there exists PE, the convergence rate is not natural to be specified as it highly depends on the excitation strength [10].

The exploitation of measurable input–output online historical data (OHD) has been shown to be feasible to relax the PE condition for parameter convergence in adaptive control [10–15]. The key idea of parameter estimation based on OHD is to utilise both OHD and information of system structures to construct a measurable feedback signal relevant to parameter estimation errors. In this way, the PE condition can be relaxed, and the convergence rate can be specified by an adaptation gain [11]. An OHD-driven model reference adaptive control method was proposed for uncertain linear time-invariant (LTI) systems, where exponential parameter convergence is obtained by a condition termed sufficient excitation (SE) which is strictly weaker than PE. The parameter estimation scheme of [11] can be interpreted as a standard least-squares estimation algorithm equipped with an LTI filter [12], whereas it faces two fatal drawbacks: (i) the setting of initial conditions is tough; (ii) noise-sensitive filtering needs to be employed.

In [10], the OHD-driven parameter estimation was extended to non-linear systems with linear-in-the-parameters (LIP) uncertainties without involving the drawbacks in the approach of [11], where finite-time parameter convergence is established under the SE condition, and robust parameter estimation is demonstrated under external disturbances or modelling errors. In [13], the approach of [10] was applied to control servomechanisms. In [14], the robustness of the approach of [10] was shown in a hybrid system framework, where the hybrid system contains a globally asymptotically stable compact set that is robust to small perturbations. Specifically, small perturbations of system data lead to only minor errors in parameter estimates. A distinctive feature of the approach of [10] is that parameter convergence is achieved at any time instant that SE exists. However, this approach has the following drawbacks: (i) the invertibility of an integral matrix needs to be checked online and the inverse matrix has to be solved when necessary, which significantly increases computational cost; (ii) the direct calculation of the actual parameter is separated from its adaptive part resulting in a discontinuous estimator, which inevitably complicates the estimator design. In [15], an OHD-based continuous parameter estimation scheme was given to guarantee exponential parameter convergence and to eliminate the drawbacks in the approach of [10]. Nevertheless, since all OHD are exploited in the SE condition, the aforementioned approaches not only yield in result in unbounded adaptation but also cannot handle parameter variations. In [16–18], OHD with forgetting factors are resorted to guarantee exact parameter estimation in adaptive control under a weaker PE condition.

This brief presents a novel parameter estimation scheme for adaptive control to eliminate the drawbacks of the existing approaches, where partial OHD are exploited to update the parameter estimate so that exponential parameter convergence is guaranteed by an interval excitation (IE) condition which is also strictly weaker than the PE condition. Compared with the full OHD-driven parameter estimation, the proposed approach contains the following advantages: (i) bounded adaptation is guaranteed as only partial OHD are exploited; (ii) the case with slow time-varying uncertainties can be handled directly; (iii) fewer parameters and variables are used resulting in a simpler parameter estimation scheme. Please refer to [19–23] for our previous results in composite learning. Note that the application of integration by parts to avoid the estimation of plant states in [19] cannot be extended to a general class of non-linear systems with mismatched uncertainties, and the extension of the proposed approach to an output-feedback case can be referred to [20]. Compared with the
approaches of [20–23], the proposed approach does not resort to the time derivation or filtering of plant states resulting in a more efficient and less noise-sensitive parameter estimation scheme. The proposed approach can be applied to enhance the performance of the classical adaptive control for industrial plants with unknown constant or slow time-varying parameters.

In this brief, Section 2 formulates the estimation problem; Section 3 revisits the approach of [15] and presents our approach; Section 4 provides numerical results; Section 5 concludes this study. Throughout this brief, \( \mathbb{R}, \mathbb{R}^+, \mathbb{R}^m \) and \( \mathbb{R}^{m \times n} \) represent the spaces of real numbers, positive real numbers, real \( n \)-vectors and real \( m \times n \)-matrices, respectively. \( \| \cdot \| \) denotes the Euclidean norm of \( \cdot \), \( I \) denotes an identity matrix, \( \Phi \) denotes a zero matrix, \( L_{\infty} \) denotes the space of bounded signals, \( \min \{ \cdot \} \) and \( \max \{ \cdot \} \) denote the minimum and maximum operators, respectively. \( \lambda_{\text{min}}(A) \) denotes the minimal eigenvalue of \( A \), and \( \sigma_{\text{min}}(A) := \sqrt{\lambda_{\text{min}}(A^T A)} \) denotes the minimal singular value of \( A \), where \( x \in \mathbb{R}^n \), \( A \in \mathbb{R}^{m \times n} \), and \( m \) and \( n \) are natural numbers. For the sake of brevity, in the following sections, the arguments of a function may be omitted while the context is sufficiently explicit.

2 Problem formulation
Consider a general class of non-linear systems with LIP uncertainties in the following form [10]:

\[
\begin{align*}
    x &= f(x, u) + \Phi(x, u)\theta \\
    y &= h(x, u)
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is a system state, \( u(t) \in \mathbb{R}^m \) is a control input, \( y(t) \in \mathbb{R}^n \) is a controlled output, \( f: \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n \), \( h: \mathbb{R}^n \mapsto \mathbb{R}^m \) and the columns of \( \Phi: \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^{n \times n} \) are known vector fields, and \( \theta \in \mathbb{R}^n \) is an unknown parameter. Note that \( \Phi \) is regarded as a non-linear regressor. In this study, we assume that \( \theta \) is constant, \( x \) is measurable, and there is an admissible control \( u \) such that the closed-loop system has uniform stability in the sense of all signals involved being of \( L_{\infty} \) [10]. Stable adaptive control designs for the system (1) with specific structures are available in many textbooks such as [24]. Let \( T_e \in \mathbb{R}^+ \) be a certain time. The following definitions are introduced for the control design [20].

**Definition 1:** A bounded signal \( \Phi(t) \in \mathbb{R}^{n \times n} \) is of PE if there exists a \( \sigma \in \mathbb{R}^+ \) such that \( \int_{-\sigma}^{\tau} \Phi(\xi)\Phi^T(\xi) d\xi \geq \sigma I, \forall \tau \geq 0 \).

**Definition 2:** A bounded signal \( \Phi(t) \in \mathbb{R}^{n \times n} \) is of IE if there exists a \( \sigma \in \mathbb{R}^+ \), \( T_e \) such that \( \int_{-t}^{T_e} \Phi(\xi)\Phi^T(\xi) d\xi \geq \sigma I, t \geq T_e \).

**Definition 3:** A bounded signal \( \Phi(t) \in \mathbb{R}^{n \times n} \) is of SE if there exists a \( \sigma \in \mathbb{R}^+ \), \( T_e \) such that \( \int_{-t}^{T_e} \Phi(\xi)\Phi^T(\xi) d\xi \geq \sigma I, t \geq T_e \).

In the above definitions, the integral for \( t < 0 \) is defined to be \( 0 \). It is intuitive to regard as an excitation strength which can be determined by \( \sigma_{\text{min}}(\Phi(t)\Phi^T(t)) \). Our objective is to design an online parameter estimation scheme for the system (1) such that the unknown parameter \( \theta \) is estimated exactly during adaptive control without involving the drawbacks in the existing OHD-driven parameter estimation schemes.

**Remark 1:** It is clear from Definitions 1–3 that: (i) the PE condition is strongest as it requires the excitation to be persistent for all time \( t \geq 0 \); (ii) the IE condition is strictly weaker than the PE condition as it only requires the excitation occurs at a certain time \( t = T_e \) which may be fulfilled during transient processes; and (iii) the SE condition is even weaker than the IE condition as the integral starts from \( t = 0 \) instead of \( t = T_e \) to fully utilise on-line information. However, in the SE condition, the exploitation of all OHD not only results in possible unbounded adaptation as the integrated matrix \( \Phi\Phi^T \) is positive-semidefinite, but also loses the flexibility to handle a slow time-varying \( \theta \).

3 Parameter estimation in adaptive control

3.1 Revisit of a recent result
This section revisits the OHD-based parameter estimation scheme in [15]. Let \( \hat{x}(t) \in \mathbb{R}^n \) and \( \hat{\theta}(t) \in \mathbb{R}^n \) denote estimates of \( x(t) \) and \( \theta(t) \), respectively. Define a prediction error \( \tilde{x}(t) := x(t) - \hat{x}(t) \) and a parameter estimation error \( \tilde{\theta}(t) := \theta - \hat{\theta}(t) \). Afterwards, a state predictor is given as follows:

\[
\dot{\hat{x}} = f(x, u) + \Phi^T(x, u)\theta_0 + k_s \tilde{x}
\]

with \( \theta_0 \in \mathbb{R}^n \) an initial estimate of \( \theta \) and \( k_s \in \mathbb{R}^+ \) a filter parameter. Subtracting (2) from (1) yields the prediction error dynamics:

\[
\dot{\tilde{x}} = -k_s \tilde{x} + \Phi^T(x, u)(\theta - \theta_0).
\]

Define an auxiliary variable \( \eta(t) \in \mathbb{R}^n \) as follows:

\[
\eta(t) := \tilde{x} - \Phi^T(t)(\theta - \theta_0)
\]

where \( \Phi_f(t) \in \mathbb{R}^{n \times n} \) is a filtered regressor generated by

\[
\Phi_f = -k_s \Phi_f + \Phi(x, u), \Phi_f(0) = 0.
\]

Making the time derivative of \( \eta \) in (4) and applying (3)–(5) to the resulting expression, one obtains

\[
\begin{align*}
\dot{\eta} &= -k_s \dot{\eta} \quad \text{with} \quad \eta(0) = \tilde{x}(0). \tag{6}
\end{align*}
\]

An excitation matrix \( \Theta(t) \in \mathbb{R}^{n \times n} \) is generated by

\[
\Theta(t) = \int_0^t \Phi_f(\tau)\Phi_f^T(\tau) d\tau, \quad \Theta(0) = 0 \tag{7}
\]

which is symmetric and positive-semidefinite as \( \Phi_f\Phi_f^T \) is symmetric and positive-semidefinite. The update law of \( \hat{\theta} \) is

\[
\dot{\hat{\theta}}(t) = \Gamma(\hat{\theta}(t) - \Theta(t)\eta(t)), \quad \hat{\theta}(0) = \Theta(t) \tag{8}
\]

in which \( \hat{\theta}(t) \in \mathbb{R}^n \) is an auxiliary variable, and \( \Gamma \in \mathbb{R}^{n \times n} \) is a positive-definite matrix of learning rates. It is shown in Theorem 4.1 of [15] that for the system (1) with the state predictor (2), if there exist constants \( \sigma, T_e \in \mathbb{R}^+ \) such that the SE condition \( \Theta(T_e) \geq \sigma I \) is given by (7) in Definition 3 is satisfied, then the parameter estimation scheme composed of (6)–(9) ensures that the parameter estimation error \( \dot{\theta}(t) \) is non-increasing for \( t \in [0, T_e) \) and exponentially converges to 0 at \( t \geq T_e \), \( \infty \) with a convergence rate not less than \( \lambda_{\text{min}}(\Gamma)\sigma \).

**Remark 2:** The parameter estimation scheme of [15] has some merits: (i) Exponential parameter convergence is achieved by the SE condition in Definition 3 which is strictly weaker than the PE condition in Definition 1; (ii) the convergence rate can be specified to be arbitrarily high subject to the excitation strength \( \sigma \) and the matrix \( \Gamma \); (iii) the noise-sensitive filtering \( (s/(s + k_s))x(t) \) with \( s \) being a complex variable is not necessary. However, since all OHD are utilised via (7) to update the parameter estimate \( \hat{\theta} \), the parameter estimation scheme of [15] is not suitable to handle a slow time-varying \( \theta \) even if PE exists. In addition, as \( \Phi_f\Phi_f^T \) is positive-semidefinite, the gain of the excitation matrix \( \Phi_f \) in (7) is monotonic increasing resulting in possible unbounded adaptation.
3.2 Novel online parameter estimation

This section presents a novel OHD-driven parameter estimation scheme to avoid the drawbacks of the existing OHD-driven parameter estimation schemes and to show that the introduction of the auxiliary variable $\eta$ in (4) is not necessary. To this end, a slightly modified state predictor which removes $\theta_i$ in (2) and sets an initial condition $\hat{x}(0) = 0$ is given as follows:

$$\dot{x} = f(x, u) + k_j \hat{x}, \quad \hat{x}(0) = x(0)$$  \hskip -1cm (10)

Subtracting (10) from (1) yields the prediction error dynamics

$$\dot{x} = -k_j \hat{x} + \Phi^T(x, u) \theta, \quad \hat{x}(0) = 0.$$  \hskip -1cm (11)

As $\hat{x}(0) = 0$, the hybrid time–frequency domain notation [25] can be applied to (11) resulting in

$$\dot{x} = \frac{1}{\sigma} \Phi^T(x, u) \theta.$$  \hskip -1cm (12)

Applying the definition of $\Phi_j$ with $\Phi_j(0) = \theta$ in (5) to (12), one gets $\Phi^T \theta$ implying

$$\Phi_j \dot{x} = \Phi \Phi^T \theta.$$  \hskip -1cm (13)

An interval-integral matrix $\Theta(t) \in \mathbb{R}^{N \times N}$ is given by

$$\Theta(t) = \int_{t_\sigma}^t \Phi_j(\tau) \Phi^T_j(\tau), \quad \Theta(0) = 0$$  \hskip -1cm (14)

in which $t_\sigma \in \mathbb{R}^+$ is an integral duration. Then, the IE condition in Definition 2 is rewritten as $\Theta(T_\sigma) \geq \sigma \Theta$ with $T_\sigma \in \mathbb{R}^+$. Usually, the epoch $T_\sigma$ that satisfies the IE condition is not unique so that the corresponding $\sigma$ can be time varying. Let $T_e$ be the first epoch that satisfies the IE condition. While the parameter $\theta$ is strictly constant, to make use of excitation information, define a current maximal excitation strength $\sigma_i(t) := \max_{\tau \in [T_\sigma, t]} |\sigma(\tau)|$, where $t_\sigma := \max_{\tau \in [T_e, t]} |\sigma(\tau)|$ denotes the epoch corresponding to $\sigma_i(t)$. An illustration of $\sigma_i$ is given in Fig. 1, where the red dash line denotes $\sigma_e$. In this case, the epoch $t_e$ is given by

$$t_e(t) = \begin{cases} t & \text{for } t \in [T_e, T_\sigma) \cup [T_\sigma, T_e) \\ T_e & \text{for } t \in [T_e, T_\sigma) \\ T_\sigma & \text{for } t \in [T_\sigma, \infty) \end{cases}$$

in which $t_e = t$ for $t < T_\sigma$. While the parameter $\theta$ is slow time-varying, as the interval integral (14) is naturally effective to handle slow time-varying uncertainties, it is wiser to set $t_e(t) = t$ and adjust the integral duration $\tau_\sigma$ according to the changing rate of $\theta$.

Now, a novel update law of $\hat{\theta}$ is presented as follows:

$$\dot{\hat{\theta}}(t) = \Gamma(\hat{\theta}(t_e(t)) - \Theta(t_e(t))) \dot{\hat{\theta}}(t), \quad \hat{\theta}(0) \in \mathbb{R}^N.$$  \hskip -1cm (15)

$$\dot{\theta}(t) = \int_{t_e(t_\sigma)}^t \Phi_j(\tau) x(\tau) d\tau, \quad \dot{\theta}(0) = \theta(0).$$  \hskip -1cm (16)

Applying (13) to (16), one gets $\dot{\theta} = \theta \theta$. Applying $\theta = \theta \theta$ to (15) yields $\dot{\theta} = -\Gamma \Theta \theta$. Thus, it follows the same proof of Theorem 4.1 in [15] that for the system (1) with the state predictor (10), if there exist $t_\sigma, c, T_\sigma \in \mathbb{R}^+$ so that the IE condition $\Theta(T_\sigma) \geq \sigma \theta$ ($\Theta$ is given by (14)) in Definition 2 is satisfied, the parameter estimation scheme (14)-(16) ensures the parameter estimation error $\dot{\theta}$ is non-increasing for $t \in [0, T_\sigma)$ and exponentially converges to $\theta$ at $t \in [T_\sigma, \infty)$ with a convergence rate not less than $\lambda_{min}(\Gamma \sigma)$.

Remark 3: In the proposed parameter estimation scheme, the moving time-window integrals in (14) and (16) are applied such that only partial OHD are exploited to update the parameter estimate $\theta$ in (15). As only latest data in a time window are utilised, the proposed approach not only ensures bounded adaptation, but also is applicable to a slow time-varying $\theta$. Thus, the limitations of the full OHD-driven parameter estimation are avoided. In addition, the proposed approach also removes the initial estimate $\theta_i$ in (2) and the auxiliary variable $\eta$ in (4), both introduced in the approach of [15], resulting in a simpler parameter estimation structure. Moreover, compared with our previous composite learning results [20–23], the distinctive feature of the proposed approach is that only the measurable signals $x$ and $u$ are required and both the time derivation of $x$ and the filtering $(s/(s + k_j))x$ are not needed resulting in a more efficient and less noise-sensitive parameter estimation scheme.

3.3 Incorporate with direct adaptive control

It is assumed that an admissible control $u$ results in the following uniformly stable closed-loop system [15]:

$$e = \lambda e + \Phi^T(x, u) \theta, \quad e(0) \in \mathbb{R}^n$$  \hskip -1cm (17)

$$\dot{\theta} = -\Gamma \Psi(x, u) e, \quad \hat{\theta}(0) \in \mathbb{R}^N$$  \hskip -1cm (18)

where $e \in \mathbb{R}^n$ denotes a tracking error, $\Psi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ denotes an augmented regressor, and $\Lambda \in \mathbb{R}^{n \times n}$ is a strictly Hurwitz matrix. Thus, given any matrix $Q \in \mathbb{R}^{n \times n}$ satisfying $Q = Q^T > 0$, there exists a unique solution $P \in \mathbb{R}^{n \times n}$ satisfying $P = P^T > 0$ for the following Lyapunov equation:

$$\Lambda^T P + P A = - Q.$$  \hskip -1cm (19)

Combining (15) with (18), one gets a composite learning law of $\dot{\theta}$ for adaptive control as follows:

$$\dot{\theta} = \Gamma (\Psi e + \kappa (\hat{\theta} - \theta))$$  \hskip -1cm (20)

with $\kappa \in \mathbb{R}^n$ a weight factor, which is the same as that in [15] except $(\hat{\theta} - \theta \theta)$ is obtained by (14) and (16) rather than (6), (7) and (9). According to the proof of Theorem 5.1 of [15], the new closed-loop system composed of (17) and (20) has exponential stability in the sense that both the tracking error $e(t)$ and the parameter estimation error $\dot{\theta}(t)$ exponentially converge to $\theta$ at $t \in [T_\sigma, \infty)$.

Remark 4: Like most adaptive control approaches such as those in all references, the proposed approach is designed to handle constant or slow time-varying parameters only, and therefore, handling fast time-varying parameters is not an objective of this study. In the proposed approach, to cancel out the influence of fast time-varying parameters, the gain of the learning rate matrix must be set to be very high resulting in high-gain learning, which increases the sampling rate needed in real-time applications and decreases robustness against measurement noise in parameter estimation. That is, the robustness to fast time-varying parameters of the proposed approach depends on high-gain learning.
4 Numerical results

4.1 Example 1: A second-order system

Consider the following model of a mass-spring system [15]:

\[
\begin{aligned}
\dot{x} &= [x, \dot{x}]^T, \\
\ddot{x} &= \begin{bmatrix} 0 & 1 \\ -x_0 & -x_0 - x_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\dot{x}} \end{bmatrix}
\end{aligned}
\]

with \( x = [x_1, x_2]^T \), \( \theta = [\theta_1, \theta_2, \theta_3]^T \), \( h(x) = x_1 \), \( f(x, u) = [x_0, u]^T \) and \( \Phi(x) = [0, \phi_1(x)] \) with \( \phi_1(x) = [-x_3, -x_1 - x_3]^T \).

Let \( y_d(t) \in \mathbb{R}^1 \) denote a desired output satisfying \( y_d(t) = x_1(t) - y_{d,1}(t) \) for \( i = 1 \) and 2. The control law \( u \) is designed as follows [15]:

\[
\begin{align*}
\dot{x} &= y_d - k_e \epsilon - \phi_1 \dot{\theta} \\
\dot{\theta} &= \Gamma(\phi_1^T P \phi_1 + x(\theta - \Theta \dot{\theta}))
\end{align*}
\]

where \( k_e = [k_e_1, k_e_2]^T \in \mathbb{R}^2 \) is a control gain, \( b = [0, 1]^T \), and \( P \) is obtained by solving (19) with \( A = [0, 1; -k_e_1, -k_e_2] \).

The performance of the proposed approach (using the update law of \( \theta \) in (20) with (14)–(16)) is compared with that of the OHD-driven adaptive control approach in [15] (using the update law of \( \theta \) in (20) with (6)–(9)), where the control laws of all approaches have the same form as (21). For simulation, set \( x(0) = [0, 0, 0]^T \).

The design parameters are set as \( k_e = [10, 2]^T \), \( \Gamma = 100 I \), \( \kappa = 10 \) and \( k_f = 10 \) for the approach of [15] and the proposed approach, and \( \tau_d = 10 \) for the proposed approach. Simulations are carried out in MATLAB software, where the solver is set as fixed-step ode 4, the step size is set as 1 ms, and the other settings are kept at their defaults.

Case 1: Constant parameter estimation. Let \( \theta = [0.1, 0.5, 1.5]^T(1 + 0.2 \sin \frac{\pi t}{3}) \) such that it is slow time-varying and \( y_d(t) = \sin t \) so that PE is met for the estimation of the slow time-varying \( \theta \). Parameter estimation trajectories by the approach of [15] and the proposed approach are depicted in Fig. 3.

For the approach of [15], due to the usage of all OHD, the parameter estimate \( \theta \) does not converge to its true value \( \theta \), and the gain of the excitation matrix \( \Theta \) is monotonically increasing (see Fig. 3a). On the contrary, for the proposed approach, an exact estimation of the slow time-varying \( \theta \) by \( \dot{\theta} \) is achieved under a limited gain of \( \Theta \) (see Fig. 3b).

Case 2: Robustness against measurement noise. To demonstrate robustness against measurement noise of the proposed approach, an AGWN channel with the same setting of that in [20] is added to the measurement of \( x \) during simulations, where the SNR of the AGWN channel is set as 50 dB, and its input signal power is set as 1 W. A performance comparison of the proposed approach with and without measurement noise is shown in Fig. 4, where the maximum of \( \| \epsilon \| \) is increased from 1.889×10^5 to 2.712×10^5 and the maximum of \( \| \dot{\theta} \| \) is increased from 1.172×10^2 to 1.392×10^2 due to the measurement noise.

Thus, the accuracy loss of the parameter estimation error \( \dot{\theta} \) is very slight under the noisy measurement and is also much less than that of the tracking error \( \epsilon \).

4.2 Example 2: A third-order system

Consider a strict-feedback uncertain non-linear system [6]

\[
\begin{align*}
\dot{x} &= [x_1, x_2, x_3]^T \\
\ddot{x} &= \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 \theta_1 \\ x_2 \theta_2 \\ (1 + x_1)u \end{bmatrix} \\
y &= x_1
\end{align*}
\]

with \( x = [x_1, x_2, x_3]^T \) and \( \theta = [\theta_1, \theta_2, \theta_3]^T \). Noting (1), one has \( h(x) = x_1 \), \( f(x, u) = [x_1, x_2, (1 + x_1)u]^T \) and \( \Phi(x) = [\phi_1(x_1), \phi_2(x_2), \phi_3(x_3)] \) with \( \phi_1(x_1) = [x_1, 0, 0, 0]^T \), \( \phi_2(x_2) = [0, 0, 0, 0]^T \) and \( \phi_3(x_3) = [0, 0, x_2, x_1]^T \).

Let \( y_d(t) \in \mathbb{R} \) be a desired output satisfying \( y_d(t) \in L_{\infty} \) for \( i = 0 \) to 3 and \( \epsilon := [e_1, e_2, e_3]^T \), where \( e_d(t) := x_1(t) - x_1(t) \) for \( i = 1 \) to 3 with \( \alpha_i(t) = y_d(t) \) and \( u(t), \alpha_i(t) \in \mathbb{R} \) being virtual control inputs. A backstepping control law \( u = \pi(k_i, e, y_d, \dot{\theta}) \) in [6] is applied to the
above system, in which $k_c \in \mathbb{R}^3$ and $y_d = [y_d, y_d, y_d, y_d]^T$. The performance of the proposed approach (using the update law of $\theta$ in (20) with (14)–(16)) is compared with those of the classical adaptive backstepping control in [6] (using the update law of $\theta$ in (18)) and the OHD-driven adaptive control approach in [15] (using the update law of $\theta$ in (20) with (6)–(9)), where the control laws of all approaches have the same form as that in [6]. The settings of the design parameters and the simulation keep the same as Section 4.1 except $x(0) = 0, k_c = [3, 3, 3]^T$ and $\Gamma = I$.

**Case 1: Constant parameter estimation.** Let $\theta = [-2, -1, 1, 2, 3]^T$ so that it is strictly constant and $y_d(t) = \sin t$ so that PE does not exist [6]. Parameter estimation trajectories by the approach of [6] and the proposed approach are depicted in Fig. 5. For the approach of [6], due to the absence of PE, two elements of the parameter estimation error $\hat{\theta}$ only converge to some constants rather than 0 (see Fig. 5a), which is consistent with the result in Fig. 1(b) of [6]. On the contrary, for the proposed approach, all elements of $\hat{\theta}$ converge to 0 under a short and weak excitation and a limited gain of the excitation matrix $\Theta$ (see Fig. 5b).

**Case 2: Slow time-varying parameter estimation.** Let $\theta = [-2, -1, 1, 2, 3]^T(1 + 0.2 \sin \frac{\pi}{250}t)$ so that it is slow time-varying and $y_d(t) = \sin \frac{1}{2}t$ so that PE exists [6] for the estimation of the slow time-varying $\theta$. Parameter estimation trajectories by the approach of [15] and the proposed approach are depicted in Fig. 6. For the approach of [15], due to the usage of all OHD, all elements of $\hat{\theta}$ do not converge to 0 at the beginning of running and become diverging as the time $t$ evolves, and the gain of the excitation matrix $\Theta$ is monotonic increasing (see Fig. 6a). On the contrary, for the proposed approach, an exact estimation of the slow time-varying $\theta$ by $\hat{\theta}$ is achieved under a limited gain of $\Theta$ (see Fig. 6b).

**Remark 5:** It follows from the theoretical derivation that the proposed approach is applicable to a class of parametric uncertain non-linear systems with any $n$th order. We select the second and third order systems as illustrative examples not only because they are from two classical literature, but also because they represent a wide class of realistic industrial systems.

**5 Conclusion**

This brief has presented a novel OHD-driven parameter estimation scheme for adaptive control, in which partial OHD are exploited to update the parameter estimate such that parameter convergence is guaranteed by the IE condition which is strictly weaker than the PE condition. The proposed approach not only eliminates the drawbacks of the existing OHD-driven parameter estimation schemes, including possible unbounded adaptation and inapplicability of slow time-varying uncertainties, but also has a simpler parameter estimation structure. The effectiveness and
superiority of the proposed approach have been verified by two illustrative examples.

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7 References


Fig. 5 Constant parameter estimation without PE in Example 2
(a) Approach of [6], (b) Proposed approach

Fig. 6 Slow time-varying parameter estimation with PE in Example 2
(a) Approach of [15], (b) Proposed approach