Efficient PID Tracking Control of Robotic Manipulators Driven by Compliant Actuators

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Abstract—This brief aims to show that a linear proportional–integral–derivative (PID) controller is theoretically valid for tracking control of robotic manipulators driven by compliant actuators. The control problem is formulated into a three-time-scale singular perturbation formula, including a slow time scale at the rigid robot dynamics, one actual fast time scale at the actuator dynamics, and another virtual fast time scale at the controller dynamics. A PID-type controller is derived to guarantee semiglobal practical exponential stability of the rigid robot dynamics, and a derivative-type controller is applied to establish global exponential stability of the actuator dynamics. Based on a state transformation to the closed-loop rigid robot dynamics and the extended Tikhonov’s theorem, it is proven that the entire system has semiglobal practical exponential stability under a proper choice of control parameters. The proposed controller is not only structurally simple and model-free resulting in low implementation cost, but also robust against external disturbances and parameter variations. The current design is only valid while the spring stiffness is relatively large compared with other parameters of the robot dynamics. Experimental results based on a single-link compliant robotic manipulator have verified effectiveness of the proposed approach.

Index Terms—Compliant actuator, flexible-joint robot (FJR), linear control, singular perturbation, trajectory tracking.

I. INTRODUCTION

Conventionally, industrial robots are driven by stiff actuators to achieve favorable position control, which possibly results in unsafety when robots and humans have physical interaction [1]. This motivates the investigation of compliant actuators [2]–[8]. A series elastic actuator (SEA) is a popular type of compliant actuators, where a spring is intentionally introduced between the motor and the load [9]. Compared with stiff actuators, the SEA has several attractive features, including smooth force transmission, low output impedance, back-driveability, shock tolerance, and energy efficiency [5]. Due to the oscillatory modes introduced, compliant actuation possibly decreases tracking accuracy and system bandwidth or even destroys closed-loop stability, which increases control difficulty for robotic applications [10].

A compliant robotic manipulator (CRM) can be formulated with other parameters of the robot dynamics. Experimental results indicate that the singular perturbation-based control has a satisfactory tracking performance with reduced control chattering and magnitude, if the spring stiffness is relatively high compared with other parameters of the FJR dynamics [16].

Existing singular perturbation-based control approaches, such as [13]–[15], have a critical stability problem, as Tikhonov’s theorem [17] invoked therein is only valid for finite time such that tracking error convergence cannot be guaranteed in theory [14]. The extended Tikhonov’s theorem [17], which is valid for infinite time, can be employed to overcome the aforementioned limitation, where this theorem has an additional condition that the rigid robot dynamics is exponentially stable. Several control strategies, such as computed torque control [18] and adaptive control [19], have been resorted to achieve exponential stability of the rigid robot dynamics. However, an accurate FJR model is necessary for computed torque control, and a strong condition named persistent excitation is required to guarantee exponential stability in adaptive control.

On the other hand, there are only a few proportional–integral–derivative (PID) controllers specified for FJRs so far [25]–[27]. A fuzzy PID controller was developed for FJRs in [25], where fuzzy rules are employed to tune PID gains. An integrated self-learning PID with a fuzzy controller was developed for FJRs in [26], in which a modified recursive least-squares algorithm is employed to tune PID gains. However, stability analysis is not provided and only set-point control is considered in [25] and [26]. Theoretical validation of PID controllers for set-point control of FJRs was studied in [27], where it is shown that a PD control action on the motor position and an integral control action on the link position are sufficient to provide semiglobal asymptotic stability of the closed-loop FJR system. It is worth noting that the theoretical validation of PID controllers for tracking control of FJRs has not been demonstrated so far.

This brief revisits PID control of CRMs from a viewpoint of singular perturbation, in which a special linear PID controller is proven to be theoretically valid for tracking control of
CRMs. The proposed control design has the following steps.

1) The control problem considered is formulated into a three-time-scale singular perturbation form, including a slow time scale at the rigid robot dynamics, one actual fast time scale at the actuation dynamics, and another virtual fast time scale at the controller dynamics.

2) A PID-type controller is derived based on approximation dynamic inversion to get semiglobal practical exponential stability of the slow rigid robot dynamics.

3) A derivative-type controller is used to obtain global exponential stability of the fast actuator dynamics.

4) A state transformation is applied to the closed-loop rigid robot dynamics that has practical exponential stability to render new dynamics that is exponentially stable.

5) The extended Tikhonov’s theorem is invoked to establish semiglobal practical exponential stability of the full system under a proper choice of control parameters.

The significance of this brief is that it not only gives a new angle and a rigorous theoretical support for PID tracking control of CRMs, but also induces a useful guide for tuning PID gains, which is undoubtedly attractive for control practitioners.

In the rest of this brief, the control problem is formulated in Section II, the PID control law is designed in Section III, experiments are provided in Section IV, and conclusions are drawn in Section V. Throughout this brief, $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^m$, and $\mathbb{R}^{m \times n}$ are the spaces of real numbers, positive real numbers, real $n$-vectors, and real $m \times n$-matrices, respectively. $||x||$ denotes the Euclidean norm of $x$, $L_\infty$ denotes the space of bounded signals, min\{}\{\}, max\{}\{\}, and inf\{}\{\} denote the minimum, maximum, and infimum operators, respectively, $B_r := \{x : ||x|| \leq r\}$ denotes the ball of radius $r$, $\partial B_r$ denotes the boundary of $B_r$, $\text{col}(x, z) := [x^T, z^T]^T$, and $C^k$ represents the space of functions for which all $k$-order derivatives exist and are continuous, where $r \in \mathbb{R}^n, x \in \mathbb{R}^n, z \in \mathbb{R}^m$, and $n, m, k$ are positive integers. Note that in Sections III and IV, the arguments of a function may be omitted while the context is sufficiently explicit.

II. PROBLEM FORMULATION

A CRM can be described by the following FJR model [11]:

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D\dot{q} + G(q) + \tau_e = K(\theta - q)$$

(1)

$$J\ddot{\theta} + K(\theta - q) = u$$

(2)

in which $q(t) := [q_1(t), q_2(t), \cdots, q_n(t)]^T \in \mathbb{R}^n$ and $\theta(t) := [\theta_1(t), \theta_2(t), \cdots, \theta_n(t)]^T \in \mathbb{R}^n$ are joint and motor angular positions, respectively, $M(q) \in \mathbb{R}^{n \times n}$ and $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ are joint and centripetal-Coriolis matrices of rigid links, respectively, $G(q) \in \mathbb{R}^n$ denotes a gravitational torque, $D\dot{q} \in \mathbb{R}^n$ denotes a viscous friction torque, $J \in \mathbb{R}^{n \times n}$ and $K \in \mathbb{R}^{n \times n}$ are inertia and stiffness matrices of SEAs, respectively, $\tau_e \in \mathbb{R}^n$ is an external disturbance, $u(t) \in \mathbb{R}^n$ is a control torque, and $n$ is the number of links. Note that $\tau_e$ is not considered in the subsequent control design. In this brief, it is assumed that $q, \dot{q}, \theta,$ and $\ddot{\theta}$ are measurable, the exact formulas of (1) and (2) are unknown, and the following properties and definitions are available for control synthesis and analysis.

Property 1 [11]: $M(q), C(q, \dot{q}),$ and $G(q)$ are of $C^1, \forall q \in \mathbb{R}^n$ and $\forall \dot{q} \in \mathbb{R}^n$, and $D, J, K$ are diagonal and constant.\(^1\)

Property 2 [11]: $M(q)$ is symmetric positive-definite, $\forall q \in \mathbb{R}^n$, and $D, J, K$ are positive-definite and diagonal.

Definition 1 [17]: $x(\epsilon) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is of the order $\epsilon$, denoted by $x(\epsilon) = O(\epsilon)$, if $\exists k, \epsilon^* \in \mathbb{R}^n$ so that $||x(\epsilon)|| \leq k\epsilon, \forall \epsilon < \epsilon^*$.

Definition 2 [20]: The origin of $\dot{x} = f(x, \epsilon)$ with $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ has practical exponential stability on $D_\in \mathbb{R}^n$ if $\exists k(x_0), r(\epsilon), \lambda, \epsilon^* \in \mathbb{R}^n$ with $x_0 = x(0)$ so that $||x(t)|| \leq \kappa(x_0) e^{-\lambda t} + r(\epsilon) \forall t \geq 0, \forall \epsilon < \epsilon^*$, and $\forall x(0) \in D_\in$. Let $\eta_\epsilon(t) := [\eta_\epsilon(1)(t), \eta_\epsilon(2)(t), \cdots, \eta_\epsilon(n)(t)]^T \in \mathbb{R}^n$ be a desired output satisfying $q_\epsilon, \dot{q}_\epsilon, \ddot{q}_\epsilon \in L_\infty$. The spring stiffness $K$ is expressed as $K = K_{0}\epsilon^2$, in which $\epsilon \in \mathbb{R}^n$ is a small constant, and $K_0 \in \mathbb{R}^{n \times n}$ is a positive-definite and diagonal matrix [14]. Both $K_0$ and $\epsilon$ are used for theoretical analysis only and will not be applied to design the control law. Define an output tracking error $e_1 := q - q_\epsilon$ and a virtual control torque $\tau := K(\theta - q)$. Applying $\tau = K(\theta - q)$ to (1) and (2), one gets

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D\dot{q} + G(q) = \tau$$

(3)

$$e^2 J\ddot{\epsilon} + K_0 \epsilon = K_0(u - J\ddot{\epsilon})$$

(4)

Substituting the expression of $\dot{\epsilon}$ by (3) into (4), one gets

$$e^2 J\ddot{\epsilon} + K_0 \epsilon = K_0(u - K_0 J M^{-1}(q)(\tau - Q))$$

where $Q(q, \dot{q}) := C(q, \dot{q}) \dot{q} + D\dot{q} + G(q)$. Multiplying both sides of the foregoing equality by $J^{-1}$ and noting the diagonal property of $K_0$ and $J$, one immediately obtains

$$e^2 \ddot{\epsilon} = K_0[J^{-1}(u - \tau) + M^{-1}(q)(Q - \tau)]$$

(5)

Let $e := \text{col}(e_1, e_2)$ and $z := \text{col}(\tau, \epsilon \dot{\epsilon})$ with $e_2 = \dot{e}_1$. Then, the system comprised of (3) and (5) is expressed as a standard singular perturbation formula

$$\begin{align*}
\dot{e} &= f_e(t, e, z), \quad e(0) = e_0 \\
\dot{z} &= g_e(t, e, z), \quad z(0) = z_0
\end{align*}$$

(6)

with $e_0, z_0 \in \mathbb{R}^{2n}$, where $f_e$ and $g_e$ are given by

$$\begin{align*}
f_e(t, e, z) &= \begin{bmatrix} e_2 \\
M^{-1}(q)(\tau - Q) - \ddot{q}_d
\end{bmatrix} \\
g_e(t, e, z) &= \begin{bmatrix} \epsilon \dot{\epsilon} \\
K_0[J^{-1}(u - \tau) + M^{-1}(q)(Q - \tau)]
\end{bmatrix}
\end{align*}$$

(7)

Due to the integral structure of (6), it is more convenient to analyze its original system composed of (3) and (5) as in [14]. It is clear that (4) at $\epsilon = 0$ has a unique isolated root\(^2\)

$$\tau^* = h_e(t, \epsilon) := u_e - J\ddot{q}$$

\(^1\)Property 1 implies each element in the matrices $M(q), C(q, \dot{q}),$ and $G(q)$ is continuously differentiable with respect to its arguments.

\(^2\)The root of (4) at $\epsilon = 0$ is equivalent to the equilibrium point of (4) at $\epsilon = 0$. The explicit time variable $t$ in $h_e(t, \epsilon)$ results from the dependance of $u_e$ on $q_\epsilon(t)$ and/or $t$ and the time-dependent $q_\epsilon(t)$. 

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freezes the variables $q$ and $\dot{q}$ is the joint acceleration. Applying $\tau = \tau^*$ to (3) at $\epsilon = 0$ and noting $\ddot{q} = \ddot{e}_1 + \ddot{q}_d$, one gets a reduced system, i.e., a quasi-steady-state model, of (6) as follows:

$$\left(M(q) + J\right)\left(\ddot{e}_1 + \ddot{q}_d\right) + Q(q, \dot{q}) = u_s. \tag{8}$$

Setting $\epsilon = 0$ in (5) and making some transformations yields

$$K_0(M^{-1}(q) + J^{-1})q = K_0 J^{-1}u_s + K_0 M^{-1}(q) Q \tag{9}$$

where $\tau = \tau^*$ and (7) is applied to get the above result. Let $t_c := t/\epsilon$ be a "stretched" time variable and $\eta := \tau - h_t(t, e)$ be a new state variable. Setting $\epsilon = 0$ freezes the variables $t$ and $e$ at 0 and $e_0$, respectively, so that $h_t(t, e)$ is constant at the fast time scale $t_c$. Substituting $\tau = \eta + h_t(t, e)$ with $h_t = 0$ into (5) and changing (5) into the fast time scale $t_c$, one has

$$\eta'' + K_0(M^{-1}(q) + J^{-1})(\eta + h_t) = K_0 J^{-1}u + K_0 M^{-1}(q) Q \tag{10}$$

with $\eta' := dq/dt_{t_c}$. Applying (9) to the above equality, one gets a boundary-layer system of (6) as follows:

$$\eta'' + K_0(M^{-1}(q) + J^{-1})\eta = K_0 J^{-1}cu_f \tag{11}$$

in which $q$ is regarded as a fixed parameter, and $\epsilon u_f := u - u_s$ is the discrepancy resulting from setting $\epsilon = 0$ in $u$. Intuitively, a composite control law is given as follows:

$$u = u_s(t, e) + cu_f(\bar{t}) \tag{12}$$

where $u_s$ and $u_f$ denote controllers for the slow reduced system (8) and the fast boundary-layer system (10), respectively. The objective of this brief is to design a model-free control law for the system given by (1) with (2) such that the joint position $q$ follows its desired signal $q_d$ closely.

### III. EFFICIENT PID CONTROL DESIGN

#### A. Control of the Reduced System

To apply the extended Tikhonov’s theorem, both the reduced system (8) and the boundary-layer system (10) should be exponentially stable [17]. This section focuses on the control of the reduced system (8), which can be rewritten as follows:

$$\ddot{e}_1 = \left(M(q) + J\right)^{-1}(u_s - Q) - \ddot{q}_d. \tag{13}$$

An ideal controller $u_s$, termed a dynamic inversion controller, can be obtained by solving

$$(M(q) + J)^{-1}(u_s - Q) - \ddot{q}_d = -K_c e \tag{14}$$

with $K_c := [K_f, K_P]$, in which $K_f, K_P \in \mathbb{R}^{n \times n}$ are positive-definite diagonal matrices of control gains.

Substituting (13) to (12), one gets the closed-loop tracking error dynamics

$$\dot{e} = \begin{bmatrix} 0 & I \\ -K_f & -K_P \end{bmatrix} e. \tag{15}$$

It follows from the positive definitiveness of $K_f$ and $K_P$ that the origin of (14) is globally exponentially stable.

The ideal controller $u_s$ acquired by solving (13) needs accurate expressions of $M(q), Q(q, \dot{q})$ and $J$. To overcome this limitation, the following fast dynamics [21]

$$\varepsilon \ddot{u}_s = \ddot{q}_d - (M(q) + J)^{-1}(u_s - Q) - K_c e \tag{16}$$

with $\varepsilon \in \mathbb{R}^+$ being a small constant is applied to generate an approximation of the ideal controller $u_s$. The solution $u_s$ of (15) is termed an approximation dynamic inversion controller, because it tends to the ideal controller $u_s$ as the decrease of $\varepsilon$ [21]. Applying (12) to (15) yields

$$\varepsilon \ddot{u}_s = -K_P e_1 - K_f e_1 - \ddot{e}_1. \tag{17}$$

Integrating both sides of (16) over $[0, t]$, one obtains an equivalent solution $u_s$ of (15) as follows:

$$u_s = -\left(K_P e_1 + K_1 \int_0^t e_1(\tau) d\tau + \dot{e}_1\right)/\varepsilon \tag{18}$$

which is in the PID form independent of $M(q), Q(q, \dot{q})$, and $J$. The system comprised of (12) and (15) can also be expressed as a standard singular perturbation form

$$\dot{e} = f_s(t, e, u_s), \quad e(0) = e_0 \tag{19}$$

$$\varepsilon \ddot{u}_s = g_s(t, e, u_s), \quad u_s(0) = u_{s0} \tag{20}$$

with $u_{s0} \in \mathbb{R}^n$, where $f_s$ and $g_s$ are given by

$$f_s(t, e, u_s) = \begin{bmatrix} (M(q) + J)^{-1}(u_s - Q) - \ddot{q}_d \\ Q(q, \dot{q}) \end{bmatrix} \tag{21}$$

$$g_s(t, e, u_s) = \ddot{q}_d - (M(q) + J)^{-1}(u_s - Q) - K_c e. \tag{22}$$

Let $(e(t, \varepsilon), u_s(t, \varepsilon))$ be a solution of the full system (18). It follows from the expression of $g_s$, that for each $(t, \varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^2$, $g_s(t, e, u_s) = 0$ has a unique isolated root:

$$u_s^* = h_1(t, e) := (M(q) + J)(\ddot{q}_d - K_c e) + Q. \tag{23}$$

Applying $u_s = u_s^*$ to (12) at $\epsilon = 0$, one gets a reduced system (14) of (18). Let $t_c := t/\epsilon$ be a "stretched" time variable and $v := u_s - h_1(t, e)$ be a new state variable. Setting $\varepsilon = 0$ freezes the variables $t$ and $e$ at 0 and $e_0$, respectively, so that $h_1(t, e)$ is constant at the fast time scale $t_c$. Applying $u_s = v + h_1(t, e)$ to (19) and $h_1 = 0$ to (15) with $\varepsilon = 0$, one obtains a boundary-layer system of (18) as follows:

$$\dot{v'} = -(M(q) + J)^{-1}v' \tag{24}$$

with $v' := dv/dt_{t_c}$, where $q$ is regarded as a fixed parameter. Let $\ddot{e}(t)$ and $\dot{e}(t_c)$ denote the solutions of the reduced system (14) and the boundary-layer system (20) with $t = 0$ and $e = e_0$, respectively. Theorem 1 shows the stability result of the reduced system (8) in regard to (6).

**Theorem 1:** For the reduced system (8) with Properties 1 and 2 driven by the PID controller $u_s$ in (17), there exists a suitably small constant $e^{**} \in \mathbb{R}^+$ such that the closed-loop system has semiglobal practical exponential stability, $\forall e \in (0, e^{**})$.

**Proof:** Consider the singularly perturbed system (18) on $$(t, e, v) \in \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^n.$$

\footnote{The linear PI control asserted in [21] is actually a PID controller as it utilizes the feedback signal $e = \text{col}(e_0, \dot{e}_1)$. Differing from the singular perturbation formulation of [21] with two time scales, the singular perturbation formulation of this brief has an additional third time scale at the actuator dynamics resulting in an increase of control difficulty.}
The conditions for utilizing the extended Tikhonov’s theorem [17, Th. 11.2] are established as follows.

1) From the expressions of $f_i$, $g_i$ in (18), and Property 1, it is obtained that for any compact subset of $\mathbb{R}^{2n} \times \mathbb{R}^n$, $f_i$, $g_i$, and their first partial derivatives with respect to $(e, u_i)$ and $(\partial g_i / \partial t)$ are continuous and bounded, $h_i(t, e)$ and $(\partial g_i / \partial u_i)$ possess bounded partial derivatives with respect to their arguments, and $[\partial f_i(t, e, h(t, e))] / \partial \phi$ is Lipschitz in $e$, uniformly in $t$.

2) As the origin of the reduced system (14) is an exponentially stable equilibrium point, there exists a Lyapunov function $V_2(e): \mathbb{R}^n \rightarrow \mathbb{R}^+$ such that $\lambda_1 \|e\|^2 \leq V_2(e) \leq \lambda_2 \|e\|^2$ and $V_2'(e) \leq -\lambda_3 \|e\|^2$ for (14) on $\mathbb{R}^{2n}$, and $(\lambda_1, \lambda_2, \lambda_3, c_3)$ is a compact subset of $\mathbb{R}^n$, where $\lambda_1$, $\lambda_2$, $\lambda_3$, and $c_3$ are some constants.

3) Because $M(q)$ and $J$ are positive-definite, uniformly in $(t, q)$ according to Property 2, the origin of the boundary-layer system (20) is an exponentially stable equilibrium point, uniformly in $(t, e)$.

Let $\Omega_e \subset \mathbb{R}^{2n}$ and $\Omega_r \subset \mathbb{R}^n$ denote the regions of attraction of the reduced system (14) and the boundary-layer system (20), respectively. As all conditions in [17, Th. 11.2] are established above, for each compact set $\Omega_e \subset \{\lambda_2 \|e\|^2 \leq \rho \} \in (0, 1)$, there is a constant $\varepsilon^* \in \mathbb{R}^+$ such that for all $t \geq 0$, $e_0 \in \Omega_e$, $u_{00} - h_r(0, e_0) \in \Omega_r$, and $e \in (0, \varepsilon^*)$, the full system (18) has a unique solution $(e(t, \varepsilon), u_r(t, \varepsilon))$ on $[0, \infty)$ and

$$
\left\{ \begin{array}{l}
ed(t, \varepsilon) = -\hat{e}(t) = O(\varepsilon)
\end{array} \right.
$$

holds uniformly for $t \in [0, \infty)$, where $O(\varepsilon)$ denotes the order of $\varepsilon$ in Definition 1. In addition, given any $t_0 > 0$, there exists a constant $\varepsilon^{**} \leq \varepsilon^*$ such that

$$
u_i(t, \varepsilon) - h_i(t, \varepsilon) = O(\varepsilon)
$$

holds uniformly for $t \in [t_0, \infty)$, as long as $e \in (0, \varepsilon^{**})$. Furthermore, as both (14) and (20) are exponentially stable, their respective solutions $\hat{e}(t)$ and $\hat{u}(t) / \varepsilon$ exist and are unique on $[0, \infty)$. Therefore, one obtains

$$
\left\{ \begin{array}{l}
\lim_{t \rightarrow \infty} \hat{e}(t) = 0,
\lim_{t \rightarrow \infty} \hat{u}(t) = 0
\end{array} \right.
$$

Applying (23) to (21) and (22), one obtains

$$
\left\{ \begin{array}{l}
\lim_{t \rightarrow \infty} \|e(t, \varepsilon)\| = 0
\lim_{t \rightarrow \infty} \|u_r(t, \varepsilon) - u^{*}(t)\| = 0
\end{array} \right.
$$

which implies practical exponential stability of the full system (18) as $\varepsilon \rightarrow 0$ according to Definition 2. Since the exponential stability of both the systems (14) and (20) is global, the region of attraction $\Omega_e \times \Omega_r$ can be arbitrarily enlarged by the decrease of $\varepsilon$ such that the stability of the full system (18) is semiglobal [22]. Moreover, the semiglobal practical exponential stability of (18) implies the semiglobal practical exponential stability of the reduced system (8), where the tracking error $e$ can be arbitrarily diminished and the region of attraction $\Omega_e$ can be arbitrarily enlarged both by the decrease of $\varepsilon$.

Remark 1: Theorem 1 indicates that the reduced system (8) driven by the PID controller $u_i$ in (17) has semiglobal stability with the performance $e(\varepsilon, \hat{e}) = O(\varepsilon)$ in (21) while the gain parameter $\varepsilon$ in (17) is smaller than a certain constant rather than arbitrarily small. The notation $\varepsilon \rightarrow 0$ is only used to show that the reduced system (8) has practical exponential stability in Definition 2 where the tracking error $e$ is dominated by $\varepsilon$. In practice, the setting of $\varepsilon$ cannot be arbitrarily small due to control saturation and noisy measurement.

B. Stability Analysis of the Entire System

The controller $u_f$ is simply chosen as a derivative-type [14]

$$
u_f = -K_D (\hat{\theta} - \hat{q}) / \varepsilon
$$

in which $K_D = K_d / \varepsilon$ with $K_d \in \mathbb{R}^{n \times n}$ is a positive-definite diagonal matrix of control gains. As $\tau = K_d (\hat{\theta} - \hat{q}) / \varepsilon^2$, $\eta = \tau - h_r(t, e)$, and $h_r(t, e)$ is constant in the boundary layer, (25) is equivalently expressed as follows:

$$
u_f = -K_d K_0^{-1} \eta.
$$

Substituting (26) into (10), one obtains a closed-loop boundary-layer system of (6) in the following form:

$$\eta'' + J^{-1} K_d \eta' + K_0 (M''(q) + J^{-1}) \eta = 0.
$$

The stability result of the boundary-layer system (10) in regard to (6) is given as follows.

Theorem 2 [14]: For the boundary-layer system (10) driven by the controller $u_f$ in (25), the origin of the closed-loop system (27) has global exponential stability, uniformly in $(t, e)$.

From the expression of $f_e$ in (6), one gets that $f_r(t, e, r) = 0$ has a unique isolated equilibrium point

$$\tau^* = \text{col}(h_r(t, e), 0)
$$

for each $(t, e) \in \mathbb{R}^+ \times D_e$, where $h_r(t, e)$ is given by (7). Applying $\tau = \tau^*$ to the first equation of (6) at $e = 0$, one gets a complete form of the reduced system of (6) as follows:

$$\dot{e} = f_e(t, e, \text{col}(h_r(t, e), 0))
$$

which is equivalent to the reduced system (8). It follows from Theorem 1 that the reduced system (29) driven by the slow controller $u_r$ in (17) has semiglobal practical exponential stability for all $e \in (0, \varepsilon^{**})$. Noting Definition 2, one has that there are some constants $\kappa(e_0), r(e), \lambda \in \mathbb{R}^+$ such that

$$\|e(t, \varepsilon)\| \leq \kappa(e_0) e^{-\lambda t} + r(e)
$$

for all $t \geq 0$, $\forall e < \varepsilon^{**}$, and $\forall e_0 \in \Omega_r$, where $r(e) \rightarrow 0$ as $e \rightarrow 0$. Since the reduced system (29) has to be exponentially stable such that the extended Tikhonov’s theorem can be invoked, a state transformation is introduced as follows $^5$:

$$e_r := \begin{cases} e - \phi_0, & \text{if } e \in B_r \\ 0, & \text{otherwise} \end{cases}
$$

$^5$It is easy to verify that $|e_r| = |e|, |\phi_r| = \inf_{x \in B_r} |e - \phi|$. The introduction of $e_r$ is only used for theoretical analysis, so that the exact values of $\phi_0$ are not required to be known.
Let $V_\epsilon(e_\epsilon): \mathbb{R}^n \mapsto \mathbb{R}^+$ denote a Lyapunov function of the reduced system (33) satisfying $\lambda_4 \|e_\epsilon\|^2 \leq V_\epsilon(e_\epsilon) \leq \lambda_5 \|e_\epsilon\|^2$ and $V_\epsilon(e_\epsilon) \leq -\lambda_6 \|e_\epsilon\|^2$ over $\mathbb{R}^{2n}$, $\Omega_{cr} \subset \Omega_r$ and $\Omega_\eta \subset \mathbb{R}^{2n}$ denote the regions of attraction of the reduced system (33) and the boundary-layer system (10), respectively, and $\{\lambda_4 \|e_\epsilon\|^2 \leq c_e\}$ denote a compact subset of $\mathbb{R}^{2n}$, where $\lambda_4$, $\lambda_5$, $\lambda_6$, and $c_e \in \mathbb{R}^+$ are some constants. From the extended Tikhonov’s theorem, for each compact set $\Omega_{cr} \subset \{\lambda_5 \|e_\epsilon\|^2 \leq \rho c_e, \rho (0,1)\}$, there is a constant $e^* \in \mathbb{R}^+$ so that for all $t \geq 0$, $e_0 - \phi_0 \in \Omega_{cr}$, $\tau_0 - \text{col}(h_\epsilon(e_0, 0), 0) \in \Omega_\eta$, and $e \in (0, e^*)$, the full system (32) has a unique solution $(e_\epsilon(t, e), \tau(t, e))$ on $[0, \infty) \cap \mathbb{R}^n$ and

$$\lim_{\varepsilon\to 0}\|e_\epsilon(t, e)\| = 0, \quad \lim_{t\to\infty} \dot{\epsilon}(t, e) = 0,$$

$$\lim_{\varepsilon\to 0}\|\phi_0\| = 0, \quad \lim_{t\to\infty} \dot{\phi}_0(t) = 0$$

(35)

holds uniformly for $t \in [0, \infty)$, In addition, given any $t_\epsilon > 0$, there exists a constant $e^{**} \leq e^*$ such that

$$\tau(t, e) - \text{col}(h_\epsilon(e_\epsilon, \phi_\epsilon) + \phi_0), 0) = \mathcal{O}(\epsilon)$$

(36)

holds uniformly for $t \in [0, \infty)$ as $\epsilon \in (0, e^{**})$. Moreover, since both (10) and (33) are exponentially stable, their respective solutions $\tilde{e}_\epsilon(t)$ and $\dot{\phi}_\epsilon(t)$ exist and are unique on $[0, \infty)$. Consequently, one obtains

$$\lim_{\epsilon\to 0, t\to\infty}\|e_\epsilon(t, e)\| = 0$$

(37)

which implies practical exponential stability of the system (32) as $\varepsilon \to 0$ from Definition 2 resulting in practical exponential stability of the system (6) as $(e, \varepsilon) \to 0$. Because the stability results of the systems (10) and (33) are semiglobal and global, respectively, the region of attraction $\Omega_{cr} \times \Omega_\eta$ can be arbitrarily enlarged by the decrease of $\varepsilon$, so that the stability of the system (6) is semiglobal [22]. Consequently, the CRM system composed of (1) and (2) driven by the composite control law (11) achieves semiglobal practical exponential stability as $(e, \epsilon) \to 0$ for all $\varepsilon \in (0, e^{**})$ and $\epsilon \in (0, e^{**})$.\[ \square \]

**Remark 2:** Theorem 3 indicates that under a proper selection of the gain parameter $\varepsilon$ in (17), the system (1) with (2) driven by the composite control law composed of (11), (17), and (25) has semiglobal stability with the performance $e_\epsilon(t, e) - \tilde{e}_\epsilon(t) = \mathcal{O}(\epsilon)$ in (35) if the spring stiffness $K$ is relatively large, rather than arbitrarily large, so that a properly small $\varepsilon$ is possible to be determined for theoretical analysis. The notation $(e, \epsilon) \to 0$ is only used to show that the entire system (1) with (2) has practical exponential stability in Definition 2, where the tracking accuracy is dominated by the gain parameters $\varepsilon$ in (17) and $K_D$ in (25) and the spring stiffness $K$ in (1). In practice, the settings of $\varepsilon$ and $K_D$ cannot be arbitrarily large as they are subject to control saturation and measurement noise. Hence, the tracking error $e$ is unable

$^6$The selection of $\varepsilon$ has a certain degree of freedom from $K = K_D/e^2$. 

Fig. 1. Illustration of the state transformation of $e$. 

with $\phi_0 := \arg \inf_{\phi \in \Omega_\eta} \|e - \phi\|$. The definition of $e_\epsilon$ on $\mathbb{R}^2$ with $e = [e_1, e_2]^T$ is shown in Fig. 1, where the length of the red dashed line denotes $\|e_\epsilon\|$. As each $e \not\in B_r$ corresponds to a unique $\phi_0$ independent of $t$, one has $\phi_0 = 0$. Applying (31) with $\phi_0 = 0$ to (6) shifts $e$ to $e_\epsilon$ as follows:

$$\dot{e}_\epsilon = f_\epsilon(t, e_\epsilon + \phi_0, \tau), \quad e_\epsilon(0) = e_0 - \phi_0$$

$$e_\epsilon = g_\epsilon(t, e_\epsilon + \phi_0, \tau, u), \quad \tau(0) = \tau_0$$

(32)

where its boundary-layer system is (10), and its reduced system is obtained by applying (31) to (29) as follows:

$$\dot{e}_\epsilon = f_\epsilon(t, e_\epsilon + \phi_0, \text{col}(h_\epsilon(t, e_\epsilon + \phi_0), 0)).$$

(33)

Let $(e_\epsilon(t, e), \tau(t, e))$ be a solution of the full system (32), and $\tilde{e}_\epsilon(t)$ and $\dot{\phi}_\epsilon(t)$ be the solutions of the reduced system (33) and the boundary-layer system (10) with $t = 0$ and $e = e_0$, respectively. From the definition of $\phi_0$, one gets $\|\phi_0\| = r$ and $\|e_\epsilon(t)\| = \|e_\epsilon(t) - \phi_0\|$. Applying these results to (30) yields

$$\|e_\epsilon(t) - \phi_0\| \leq \kappa(\epsilon_0)e^{-\lambda t}.$$ 

Noting $e_\epsilon = e - \phi_0$ and $e_\epsilon(0) = e_0 - \phi_0$, one gets

$$\|e_\epsilon(t)\| \leq \kappa(e_0 + \phi_0)e^{-\lambda t},$$

(34)

which indicates that the origin of the reduced system (33) is a semiglobally exponentially stable equilibrium point. Now, it is ready to show the stability result of the entire system.

**Theorem 3:** For the system (1) with (2) under Properties 1 and 2 driven by the control law (11) with $u_\varepsilon$ given by (17) and $u_f$ given by (25), there exist suitably small constants $\epsilon^{**}, \epsilon^{**} \in \mathbb{R}^+$ so that the closed-loop system has semiglobal practical exponential stability, $\forall \varepsilon \in (0, e^{**})$ and $\forall e \in (0, e^{**})$.

**Proof:** Consider the singularly perturbed system (32) on

$$(t, e_\epsilon, \tau - \text{col}(h_\epsilon(t, e), 0)) \in \mathbb{R}^+ \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}.$$ 

It follows from Theorems 1 and 2 that the exponential stability of the reduced system (33) is guaranteed by the controller $u_\varepsilon$ in (17) with $\varepsilon \in (0, e^{**})$ and the exponential stability of the boundary-layer system (10) is ensured by the controller $u_f$ in (25). Thus, the conditions for utilizing the extended Tikhonov’s theorem can be established by the same steps as the proof of Theorem 1 such that the details are not repeated here for saving page space.
to reach \( \mathbf{0} \) in experiments even when the CRM system (1) with (2) has a very large \( K \).

Remark 3: The classical singular perturbation theory in [17] does not provide how small \( \varepsilon \) and \( \epsilon \) are necessary in Theorem 3 to guarantee closed-loop stability. In [23], a composite control law that is similar to (11) was proposed for a class of underactuated mechanical systems, where control gains that guarantee closed-loop stability are determined based on the Lyapunov-like design. Note that both the singular perturbation theory and the compliant actuation are not involved in [23]. An optimal microalgae growth problem of the photosynthetic factory was considered in [24], where an analytical solution is obtained by the use of the singular perturbation theory. It is interesting to investigate how the approaches of [23] and [24] can be extended to this brief to determine the values of \( \varepsilon \) and \( \epsilon \) that guarantee closed-loop stability. This work is out of the scope of this brief and would be left for further studies.

Remark 4: Like most existing singular perturbation-based FJR controllers such as [13]–[15], the proposed PID controller composed of (11), (17), and (25) is only valid while the spring stiffness \( K \) is relatively high compared with \( M(q) \) and \( J \) in the CRM system (1) with (2) according to (27). Therefore, a large \( K \) may not be needed for lightweight robots. The reasons behind the demand of \( K = K_0/\epsilon^2 \) being relatively high are as follows: 1) \( K_0 \) in (27) must be suitably large to guarantee the performance of the boundary-layer system (10) and 2) \( \epsilon \) in (6) must be suitably small to guarantee the stability and performance of the CRM system (1) with (2) according to Theorem 3.

Remark 5: The demand of the spring stiffness \( K = K_0/\epsilon^2 \) being relatively high can be eliminated by using a new composite control law as follows [28]:

\[
\mathbf{u} = (I + K^{-1} K_E) \mathbf{u}_s(t, \mathbf{e}) + \epsilon \mathbf{u}_f(\mathbf{\tau}, \dot{\mathbf{\tau}})
\]

with \( \mathbf{u}_s \) being given by (17) and

\[
\mathbf{u}_f = -(K_E(\mathbf{\theta} - \mathbf{q}) + K_D(\dot{\mathbf{\theta}} - \dot{\mathbf{q}}))/\epsilon
\]

where \( K_E = K_\varepsilon/\epsilon^2 \) with \( K_\varepsilon \in \mathbb{R}^{n \times n} \) is a positive-definite diagonal matrix of control gains. However, the advantage of the above control design comes with some expenses as follows [28].

1) A reduced system resulted from the new control law (39) is no longer identical to the rigid robot dynamics (8) such that the performance of (8) is not exactly recovered under (39).
2) The slow controller \( \mathbf{u}_s \) in (39) is multiplied by \( (I + K^{-1} K_E) \) resulting in a higher control gain.
3) The new control law (39) requires the knowledge of \( K \).
4) A large \( K_E \) in (40) may cause limit cycles in the steady state without control saturation.

Remark 6: The parameters selection in the proposed control law (11) with (17) and (25) follows the following rules.

1) \( K_P \) and \( K_I \) in (17) can be determined based on desired poles of the nominal system (14).
2) \( K_D \) in (25) can be decreased to avoid high-gain control while the spring stiffness \( K \) is relatively high.

3) Decreasing \( \epsilon \) in (17) diminishes the difference between the performances of the singularly perturbed system (6) and its nominal counterpart (14), but \( \epsilon \) that is too small may lead to an impractical high control gain.

IV. EXPERIMENTAL STUDIES

A CRM driven by an upgraded version of the SEA presented in [5] was set up to carry out experiments (see Fig. 2), where the upgraded SEA is mainly comprised of a servomotor equipped with a rotary encoder, a set of linear springs, a ball screw, and two potentiometers. The motion from the motor is first transmitted to the ball screw through a coupler, which converts the rotatory motion of the shaft to linear motion of the ball screw nut. The motion of the nut is transmitted to an output carriage via the linear springs, and the output carriage drives the robot joint through a pair of cables. The rotary encoder measures the motor angle, one linear potentiometer measures the displacement of the linear springs, and another rotary potentiometer measures the joint angle. In this setup, the mass of the link \( m = 0.20 \) kg, the length of the link \( l = 0.35 \) m, and the spring stiffness \( K = 2.4 \times 10^4 \) N/m [5, Table I]. Note that all these parameters are not required to be known in the proposed control design.

The experimental device consists of the SEA-driven CRM, a dSPACE hardware kit, an Elmo Harmonica 12/60 motor driver with a maximum power output 200 W and a continuous output current 5 A, and a personal computer. The real-time controller is realized by the dSPACE hardware kit that includes a DS1007 PPC Processor Board, a DS3002 Incremental Encoder Interface Board, a DS2102 D/A Board, and a DS2002 A/D Board. The noise level of the joint angle measurement is about 0.03°, the A/D resolution is 16 bit, and the sampling frequency is 1000 Hz. The values of the control parameters are set as \( K_P = 1 \), \( K_I = 0.5 \), \( \varepsilon = 0.1 \) in (17), and \( K_D = 0.5 \) in (25).

Let \( q_d = (\pi/10) \sin(2\pi t) \) be a baseline desired output with an amplitude 18° and a frequency 1 Hz. Control trajectories are shown in Fig. 3(a), where \( e_D := q - \theta \) denotes a deflection caused by spring deformation, and \( e_{\max} := \max_{t \geq 0} ||e(t)|| \) denotes a tracking accuracy. It is observed that both the joint
position $\mathbf{q}$ and the motor position $\theta$ closely follow the desired output $\mathbf{q}_d$, the tracking accuracy is $\varepsilon_{\text{max}} = 0.2313^\circ$ (1.29%), and the deflection $e_D$ is within $0.2021^\circ$ (1.12%). To verify the robustness of the proposed approach, the robotic end-effector is held by a human hand to generate an external disturbance $\tau_e$ in (1). Control trajectories in the presence of $\tau_e$ are shown in Fig. 3(b), where the tracking accuracy degrades to $\varepsilon_{\text{max}} = 0.7036^\circ$ (3.90%), and the control torque $u$ keeps within 0.05 N.s.

To demonstrate the high-amplitude tracking performance of the proposed approach, the amplitude of $\mathbf{q}_d$ is increased to $36^\circ$. Control trajectories of this case are shown in Fig. 4, where the tracking accuracy is $\varepsilon_{\text{max}} = 0.7036^\circ$ (3.90%), and the control torque $u$ keeps within 0.05 N.s.

V. CONCLUSION

In this brief, we have proven that a specifically designed linear PID controller is theoretically valid for tracking control of CRMs. This brief not only provides a new angle and a rigorous theoretical support for PID tracking control of CRMs, but also induces a useful guide for tuning PID gains. Experimental results have shown a favorable tracking performance of this approach. The experiments can be easily extended to CRMs.

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with $n$-degree of freedoms from Theorem 3. It is worth noting that the current design is only valid for the cast that the spring stiffness is relatively high compared with other parameters of the robot dynamics. In addition, the tracking accuracy of the proposed PID controller is restricted by control gains and spring stiffness due to the nature of practical exponential stability in Theorem 3. How to address this issue is an interesting topic for further studies. Please refer to [29] for the comparison of experimental results between the current design and the adaptive backstepping design.

REFERENCES


