Adaptive fuzzy PD control with stable $H^\infty$ tracking guarantee

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For indirect adaptive fuzzy $H^\infty$ tracking control (AFHC) of perturbed uncertain nonlinear systems, sliding-mode control (SMC) compensation usually has to be applied to ensure stability and $H^\infty$ robustness of the closed-loop system. We prove that indirect AFHC without SMC compensation is sufficient to guarantee stable $H^\infty$ tracking under given initial conditions and parameter constraints. The control structure only includes an indirect adaptive fuzzy control term and a proportional derivative (PD) control term. A certainty equivalent control law is slightly modified such that both a lumped perturbation and adaptive laws are independent of the PD control term. This modification is significant since it not only plays a key role in stability analysis, but also alleviates some drawbacks of existing AFHC approaches for practical applications. An illustrative example has been provided to verify correctness of the theoretical result.

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1. Introduction

Adaptive approximation-based control (AAC) using fuzzy systems or neural networks is effective for handling functional uncertainties in nonlinear systems [1] and has kept great attraction in recent years, where some latest related results can be referred to [2–20]. This study is focused on the control of perturbed uncertain nonlinear systems. For simplifying illustration, consider a class of single-input single-output (SISO) perturbed affine nonlinear systems in the Brunovsky canonical form [21]:

$$
\begin{align*}
\dot{x}_i &= x_{i+1}(i = 1, 2, \ldots, n - 1) \\
\dot{x}_n &= f(x) + g(x)u + d(t) \\
y &= x_1
\end{align*}
$$

where $x(t)=[x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is a state variable, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are a control input and a system output, respectively, $f(x), g(x): \mathbb{R}^n \to \mathbb{R}$ are unknown functions, and $d(t) \in \mathbb{R}$ is an unknown external disturbance. Typically, AAC includes two different schemes, namely direct and indirect schemes. In the direct scheme, a single approximator is applied as a controller to approximate an ideal control law, whereas in the indirect scheme, two approximators are implemented as estimation models to approximate the functions $f$ and $g$, respectively. In practice, AAC systems can usually be influenced by various perturbations such as approximation errors, external disturbances and unmodelled dynamics, which cannot be handled by function approximation and can degrade control performances or even destroy system stability [31].

Proportional derivative (PD) control is a popular method for suppressing the aforementioned perturbations [21–40]. It has been shown in [21] that adaptive fuzzy $H^\infty$ tracking control (AFHC), which is composed of an indirect adaptive fuzzy control (AFC) term and a PD control term, guarantees that the $L_2$-gain from a lumped perturbation $w_0$ (including optimal approximation errors) to tracking errors is not larger than a prescribed attenuation level. Usually, $g$ must be known in the direct AFHC [21–23]. This limitation was relaxed by applying some constants to replace $g$ during control synthesis in [24–26]. Yet, such a treatment greatly increases the gain of $w_0$. The indirect AFHC does not have any special requirement on $g$ [21]. However, the classical indirect AFHC [21] and some of its variations such as those in [27–30] are subjected to several fundamental limitations as follows [39,40]: 1) standard $H^\infty$ robustness cannot be guaranteed since $w_0$ relies on $u$; 2) there is a critical stability problem because a stability condition may not be satisfied even if optimal approximation errors are small. Sliding-mode control (SMC) compensation with known certain plant bounds or adaptive bounding techniques is an effective method for tackling these limitations [31–38]. However, this method results in some side effects as follows: 1) more priori knowledge of the plant has to be known; 2) there is a trade-off between chattering at $u$ and tracking accuracy; 3) implementation cost can be increased due to the additional SMC term. To our knowledge, how to tackle
the drawbacks of the classical indirect AFHC without SMC compensation is still an open question. In this paper, a novel methodology of indirect AFHC is proposed to guarantee stable $H^\infty$ tracking without SMC compensation. The design procedure is as follows: Firstly, the certainty equivalent control law is slightly modified such that both a lumped perturbation and adaptive laws are independent of the PD control term; secondly, a domain of fuzzy approximation is determined under given initial conditions; thirdly, parameter constraints that guarantee system stability are obtained by the Lyapunov synthesis. It is proven that the closed-loop system achieves stable $H^\infty$ tracking in the sense that all involved signals are uniformly bounded and tracking errors converge to a small neighborhood of zero dominated by control parameters. Compared with existing indirect AFHC works, the contributions of this study include: 1) a slightly modified certainty equivalent control law is proposed to facilitate control analysis; 2) the proposed AFHC is proven to be sufficient to ensure stable $H^\infty$ tracking without SMC compensation.

The rest of this paper is organized as follows: The problem considered is formulated in Section 2; the stable AFHC is designed in Section 3; an illustrative example is given in Section 4; conclusions are summarized in Section 5. Throughout this paper, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$ and $\mathbb{R}^{nm}$ denote the spaces of real numbers, positive real numbers, real $n$-vectors and real $n \times m$-matrices, respectively, $|| \cdot ||$ and $|| \cdot ||_\infty$ denote the Euclidean-norm and $\infty$-norm, respectively, $L_2$ and $L_\infty$ denote the spaces of square-integrable and bounded signals, respectively, $\lambda_{\min}(-\cdot)$ and $\lambda_{\max}(\cdot)$ are the functions of minimal and maximal eigenvalues, respectively, $\min(-\cdot)$, $\max(-\cdot)$ and $\sup(-\cdot)$ are the operators of minimum, maximum and supremum, respectively, $\text{col}(a, b)=[a^T, b^T]^T$, and $C^k$ is the space of functions whose $k$-order derivatives all exist and are continuous, where $a, b \in \mathbb{R}^n$, and $n, m$ and $k$ are positive integers.

2. Problem formulation

2.1. System description

Revisit the system (1) with $f$ and $g$ being of $C^1$, and rewrite it in the following compact form:

\[
\begin{align*}
\dot{x} &= A x + b f(y) + g(y)u + d(t) \\
y &= c^T x
\end{align*}
\]

where $A$, $b$ and $c$ are given by

\[
a = \begin{pmatrix} 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{pmatrix},
b = \begin{pmatrix} 0 \\
0 \\
0 \\
0 \\
1 \end{pmatrix}, c = \begin{pmatrix} 1 \\
0 \\
0 \\
0 \\
0 \end{pmatrix}
\]

Let $y_d(t) \in \mathbb{R}$ denote a desired output, $e_1(t)=y(t)-y_d(t)$ denote an output tracking error, $y(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_M(t) \end{bmatrix} \in \mathbb{R}^M$, $y_d(t)=\begin{bmatrix} y_d^1(t) \\ y_d^2(t) \\ \vdots \\ y_d^M(t) \end{bmatrix} \in \mathbb{R}^M$, and $e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_M(t) \end{bmatrix} \in \mathbb{R}^M$. Then, the following assumptions are made [37].

Assumption 1. There exist constants $g_0, d \in \mathbb{R}^+$ to satisfy $|g(x)| > g_0$ and $|d(t)| \leq d$. Without loss of generality, it is assumed that $g(x) > g_0$.

Assumption 2. It has $y_d^{i}(t) \in L_\infty$ for $i = 0, 1, \ldots, n$.

Select $k = [k_0, k_1, \ldots, k_M]^T \in \mathbb{R}^M$ such that $A_m(A - bk)^k \in \mathbb{R}^{mxn}$ is strictly Hurwitz. Then, for any given $Q \in \mathbb{R}^{mxn}$ satisfying $Q = Q^T$, 0, a unique solution $P \in \mathbb{R}^{nxn}$ satisfying $P = P^T \succ 0$ exists for the following Riccati-like equation:

\[
A^T \tilde{P} + \tilde{P}A + Q - \tilde{P}b \left( \frac{2}{r} - \frac{1}{r^2} \right) b^T \tilde{P} = 0
\]

with $r, \alpha \in \mathbb{R}^+$ if and only if $2\tau^2 > r$ [21]. Making $2\tau^2 = r$, one also obtains the following Lyapunov equation:

\[
A^T \tilde{P} + \tilde{P}A + Q - \tilde{P}b \left( \frac{2}{r} - \frac{1}{r} \right) b^T \tilde{P} = 0
\]

(4)

Remark 1. In this study, only the class of SISO affine nonlinear systems in (1) or (2) under Assumptions 1 and 2 is considered for the simplification of illustration. In Assumption 1, $|g(x)| > g_0$ and $\|d(t)\| \leq d$ imply the controllability of (1) and the boundedness of $d(t)$, respectively. In Assumption 2, $y_d, y_0, \ldots, y_d^M$ being of $L_\infty$ ensures the boundedness of $y_d$ and the existence of an ideal control law for (1). Similar theoretical results of this paper are possible to be derived for a more general class of multi-input multi-output nonlinear systems [37] and a special class of time-delayed nonlinear systems [41] under certain conditions.

2.2. Function approximation

If $f$ and $g$ are known a priori and $d(t) = 0$ in the system (1), then the following ideal control law:

\[
u^* = (-f(x) + v)(g(x)) \tag{5}
\]

with $v = y_d^{(2)} - k^e \in \mathbb{R}^n$ makes the corresponding closed-loop dynamics $\dot{e} = e$ globally and exponentially stable [1]. Since $f$ and $g$ are unknown and $d \neq 0$ in (1) in this study, the following $C^1$ linearly parameterized fuzzy systems [1]:

\[
\begin{align*}
\dot{\hat{f}}(x, \hat{\theta}_j) &= \hat{\theta}_{j1} \xi(x) = \sum_{j=1}^{M} \hat{\theta}_{j1} \xi_j(x) \\
\dot{\hat{g}}(x, \hat{\theta}_j) &= \hat{\theta}_{j2} \xi(x) = \sum_{j=1}^{M} \hat{\theta}_{j2} \xi_j(x)
\end{align*}
\]

are applied to approximate $f$ and $g$ in (5), respectively, where $\xi: \mathbb{R}^n \mapsto \mathbb{R}^m$ satisfying $|| \xi || \leq \phi$ is a vector of fuzzy basis functions, $\hat{\theta}_j = [\hat{\theta}_{j1}, \hat{\theta}_{j2}, \ldots, \hat{\theta}_{jM}]^T \in \mathbb{R}^M$ and $\hat{\theta}_k = [\hat{\theta}_{k1}, \hat{\theta}_{k2}, \ldots, \hat{\theta}_{kM}]^T \in \mathbb{R}^M$ are vectors of adjustable parameters, $\phi \in \mathbb{R}^+$ is a constant, and $M$ is the number of fuzzy rules. To facilitate presentation, define $1$

\[
\begin{align*}
Q &= \{ e \xi^T e/2 \leq \xi \}, \\
Q_\Delta &= \{ y_d \xi^T y_d \leq m_d \}, \\
D &= \{ xx^T = e + y_d, y_d \in Q_d, e \in Q_e \}, \\
\Omega &= \{ \hat{\theta}_j - m_{\hat{\theta}} \leq \hat{\theta}_j \leq m_{\hat{\theta}}, j = 1, \ldots, M \}, \\
\Omega_\Delta &= \{ \hat{\theta}_j \in \Omega, \hat{\theta}_j \in \Omega_\Delta \},
\end{align*}
\]

where $D_x = \Omega_x \times \Omega_x$ is a domain of fuzzy approximation, and $C_x, m_{\hat{\theta}}, m_{\hat{\theta}}$ are preassigned parameters. Define optimal approximation errors $w_y$ and $w_\Delta$ as follows:

\[
\begin{align*}
w_y(x) &= f(y) - \hat{f}(x, \hat{\theta}_k), \\
w_\Delta(x) &= g(y) - \hat{g}(x, \hat{\theta}_k)
\end{align*}
\]

(7)

where $\xi_k^T$ and $\xi_k^T$ are vectors of optimal parameters given by

\[
\begin{align*}
\xi_k^T &= \arg \min_{\theta \in \Omega_\Delta} \sup_{x \in D_x} | f(y) - \hat{f}(x, \hat{\theta}_k) |, \\
\xi_k^T &= \arg \min_{\theta \in \Omega_\Delta} \sup_{x \in D_x} | g(y) - \hat{g}(x, \hat{\theta}_k) |
\end{align*}
\]

2.3. Revisit of previous results

A certainty equivalent control law with respect to (5) is presented as follows [21]:

\[
\begin{align*}
A^T \tilde{P} + \tilde{P}A + Q - \tilde{P}b \left( \frac{2}{r} - \frac{1}{r^2} \right) b^T \tilde{P} = 0
\end{align*}
\]

$\footnote{In mathematics, a compact set is a closed and bounded subset of an Euclidean space $\mathbb{R}^k$ for any $k = 1, 2, 3, \ldots$.}$
where $\omega_l$ denotes an indirect AFC part, and $u_c$ denotes a PD control part that is essential for achieving the $H^\infty$ tracking performance as shown in [21]. Applying (5)–(8) to (2), one obtains closed-loop tracking error dynamics as follows:

$$\dot{e} = A_e \dot{e} + b \theta_d^T \xi(x) + \hat{\theta}_d^T \xi(x)u + w_s - e^T P b r$$

(9)

with $\hat{\theta}_d^T = \hat{\theta}_d - \hat{\theta}_d$ and $\hat{\theta}_d = \hat{\theta}_d - \hat{\theta}_d$, in which $w_s$ is a lumped perturbation in [21] given by

$$w_s(x, t) := w_l(x) + w_p(x)u + d(t).$$

(10)

Adaptive laws of $\hat{\theta}_d$ and $\hat{\theta}_d$ in [21] are given as follows:

$$\hat{\theta}_d = \text{proj}(\hat{\theta}_d^T \xi(x)e^T P b)$$

$$\hat{\theta}_d = \text{proj}(\hat{\theta}_d^T \xi(x)e^T P b u)$$

(11)

where $\gamma_1, \gamma_2 \in \mathbb{R}^+$ are learning rates, and $\text{proj}(\bullet) = [\text{proj}(\bullet_1), ..., \text{proj}(\bullet_n)]^T$ is a projection operator given by [11]

$$\text{proj}(\bullet) :=
\begin{cases}
0 & \text{if } \hat{\theta}_d = \hat{\theta}_d \text{ and } \bullet < 0 \\
0 & \text{if } \hat{\theta}_d = \hat{\theta}_d \text{ and } \bullet > 0 \\
\bullet & \text{otherwise}
\end{cases}$$

Combining the definitions of $\Omega_l$ and $\Omega_p$ with the above projection operator, one has that if $\hat{\theta}_d$ and $\hat{\theta}_d$ are in the interiors of $\Omega_l$ and $\Omega_p$, respectively or on the boundaries but moving toward the insides of $\Omega_l$ and $\Omega_p$, respectively, then the projection operator is not effective. Otherwise, if $\hat{\theta}_d$ and $\hat{\theta}_d$ are on the boundaries but moving toward the outsides of $\Omega_l$ and $\Omega_p$, respectively, then $\hat{\theta}_d$ and $\hat{\theta}_d$ are projected to the boundaries of $\Omega_l$ and $\Omega_p$, respectively.

Choose a Lyapunov function candidate

$$V(z) = \frac{1}{2} e^T P e + \frac{1}{2} \gamma_1 \hat{\theta}_d^T \hat{\theta}_d + \frac{1}{2} \gamma_2 \hat{\theta}_d^T \hat{\theta}_d$$

(12)

with $z = \text{col}(e, \hat{\theta}_d, \hat{\theta}_d)$ for the closed-loop system constituted by (8)–(11). The following lemma indicates the performance of the previous indirect AFHC in [21] for the system (1).

Lemma. [21]: Consider the system (1) with Assumptions 1 and 2 driven by the control law comprised of (8) and (11), where $P$ in (8) is obtained by solving (3). If $\hat{\theta}_d(0) \in \Omega_l$ and $\hat{\theta}_d(0) \in \Omega_p$, then the closed-loop system achieves the following $H^\infty$ tracking performance:

$$\int_0^T e^T(t)Q e(t) dt \leq 2 V(0) + \rho^2 \int_0^T w_s^2(t) dt$$

(13)

$\forall T \in [0, \infty)$ and $\forall w_s \in L_2(0, T)$ which implies the $L_2$-gain from $w_s$ to $e$ is not larger than the prescribed attenuation level $\rho$ while $V(0) = 0$.

Remark 2. The fundamental limitations of the indirect AFHC in [21] and some of its variations such as those in [27–30] are demonstrated as follows:

- The performance (13) cannot guarantee standard $H^\infty$ robustness since a small $\rho$ also leads to a large $w_s$ (using (8) and (10) and $2\rho^2 \geq \gamma$) which may cancel out the effect of $\rho$ [39]:
- The system has a critical stability problem as a stability condition may not be ensured even if both $w_s$ and $w_p$ are small.$^2$

The SMC compensation with known certain plant bounds or adaptive bounding estimates [32–35,37,38,36] is the only effective method for tackling these limitations so far. Nevertheless, this method results in some side effects as discussed in Section 1.

3. Stable adaptive fuzzy control

3.1. Closed-loop system dynamics

This section shows that the drawbacks of the indirect AFHC manifested in Remark 2 can be avoided by a slight modification on the certainty equivalent control law (8) as follows:

$$u = (-\hat{f}(\hat{x}) + V)(\hat{x}) - e^T P b \dot{r}$$

(14)

with $r = 2\rho^2$, where the PD control term $u_c$ is independent of $\hat{g}$. From the $u_l$ part in (14) and the definition of $v$, one obtains $y_\bar{u} = k_l + \hat{g}(\hat{x}) + \hat{g}(\hat{x})u_l$.

Subtracting the above equality from the second line of (1) and noting $e = x - \bar{x}_l$ and $u = u_l + u_c$, one obtains a new closed-loop tracking error dynamics as follows:

$$\dot{e} = A_e \dot{e} + b \theta_d^T \xi(x) + \hat{\theta}_d^T \xi(x)u + w_s - g(x)e^T P b r$$

(15)

where $w_s$ is a new lumped perturbation given by

$$w_s(x, u_l, t) := w_l(x) + w_p(x)u_l + d(t).$$

(16)

It is worth noting that differing from the lumped perturbation $w_s$ in (10), the lumped perturbation $w_s$ in (16) is independent of the PD control term $u_c$. From (14) and the definitions of $v$ and $e$, one gets $u_l = u_l(e, y_\bar{u}, \hat{\theta}_d, \hat{\theta}_d)$. Let a possible upper bound of $w_s$ be

$$\bar{w}_s := \max_{e, u_l} \{w_s(e, u_l, t) : e \in \Omega_e, \bar{u}_l \in \Omega_{u_l}, \hat{\theta}_d \in \Omega_{\hat{\theta}_d}, \hat{\theta}_d \in \Omega_{\hat{\theta}_d}\}$$

(17)

with $\bar{w}_s = \max_{e, u_l} \{\{w_s(e, u_l)\} : \bar{u}_l = \max \{u_l(e, y_\bar{u}, \hat{\theta}_d, \hat{\theta}_d)\}$, in which the maximization is done over all $e \in \Omega_e, y_\bar{u} \in \Omega_{u_l}, \hat{\theta}_d \in \Omega_{\hat{\theta}_d}$ and $\hat{\theta}_d \in \Omega_{\hat{\theta}_d}$.

Remark 3. It should be mentioned that the possible bound of $w_s$ in (17) obtained under $e \in \Omega_e$ is just used for establishing the stability condition in Theorem 1 of Section 3.2 rather than acts as a stability condition directly. In fact, regarding $e \in \Omega_e$ as a stability condition is improper since $e \in \Omega_e$ implies the boundedness of the system (1). The approach of this study related to the specification of both initial conditions and control parameters is similar to that of [42], where the proof of Theorem 1 is inspired by the concepts of local attraction and positively invariant sets therein.

3.2. Control synthesis and analysis

Adaptive laws of $\hat{\theta}_d$ and $\hat{\theta}_d$ are redesigned as follows:$^3$:

$$\dot{\hat{\theta}_d} = \text{proj}(\gamma_{\theta} e^T P b)$$

$$\dot{\hat{\theta}_d} = \text{proj}(\gamma_{\theta} e^T P b u)$$

(18)

(footnote continued)

$^3$ Please refer to the stability condition (26) and Remark 5 in the subsequent contents for better understanding this stability problem. Specifically, if $w_s$, an upper bound of $w_s$ in [10], is applied to replace $w_s$ in (26), then because decreasing $\rho$ also increases $\bar{w}_s$, the last term of (26) cannot be diminished by the decrease of $\rho$ such that the inequality (26) may not be guaranteed.
\[
\begin{align*}
\hat{\theta}_t &= \text{proj}_{\mathcal{U}}(\xi_t)(e^T P b_t) \\
\hat{\theta}_g &= \text{proj}_{\mathcal{U}}(\xi_t)(e^T P b u_t)
\end{align*}
\] (18)
which are independent of the PD control term \(u_t\). To facilitate presentation, define two constants \(c_f\) and \(c_g\) as follows:
\[
\begin{align*}
c_f &= \max \left\{ \frac{\theta^T}{\theta^T \theta^T / 2(\gamma_f)^2} \right\} \\
c_g &= \max \left\{ \frac{\theta^T}{\theta^T \theta^T / 2(\gamma_g)^2} \right\}
\end{align*}
\] (19)
Then, let a compact set that includes all possible \(e(0)\) be
\[
\Omega_{e0} = \{ e(0) P e / 2 \leq c_{e0} \}
\]
with \(c_{e0} < c_e\) and \(\mathcal{D}_{e0} = \mathcal{D}_e \times \mathcal{D}_{e0}\). Choose the Lyapunov function candidate in (12) for the closed-loop system (15) with (18). The following theorem shows the main results of this study.

**Theorem 1.** Consider the system (1) under Assumptions 1 and 2 driven by the control law (14) with (18), where \(P\) in (14) is obtained by solving (4). For any given \(\xi(0) \in \mathcal{D}_{\xi}, \hat{\theta}_0(0) \in \hat{\mathcal{D}}, \hat{\theta}_g(0) \in \hat{\mathcal{D}}_g, \mathcal{Y}_e(t) \in \Omega_{e0}\) and an approximation domain \(\mathcal{D}_e\) subjected to \(c_e \geq c_{e0} + c_f + c_g\) (20)
then exist suitably large PD control gain \(1/r\) in (14), learning rates \(\gamma_f\) and \(\gamma_g\) in (18) and \(\min (Q)\) in (4) to satisfy (26) and (28) such that the closed-loop system achieves:

1. Stability in the sense that \(\xi(t) \in \mathcal{D}_e, \hat{\theta}_t(t) \in \hat{\mathcal{D}}_e, \hat{\theta}_g(t) \in \hat{\mathcal{D}}_g\) and \(u(t) \in \Omega_u\) in (27), \(t \geq 0\);
2. The following H∞ tracking performance:
\[
\int_0^T e^T(t) Q e(t) dt \leq 2V(0) + \rho^2 \int_0^T w^2(t) dt
\] (21)
with \(w = \omega_{e0} / \sqrt{2}\), \(\forall T \in [0, \infty)\) and \(\forall w \in L_2[0, T]\), which implies that the \(L_2\)-gain from \(w\) to \(e\) is not larger than the prescribed attenuation level \(\rho\) while \(V(0) = 0\);
3. The convergence of \(e(t)\) to a small neighborhood of zero.

**Proof.** Firstly, differentiating (12) along (15) with respect to time \(t\) and applying (4) to the resulting expression, one obtains
\[
\begin{align*}
V &= -e^T(t) Q e(t) / 2 + e^T(t) P b w(t) / 2 - g_0(e^T P b)^2 / r \\
&\quad + e^T(t) \dot{\theta}_t(t)(\xi(t) + \dot{\theta}_t(t) u(t)) - \dot{\theta}_g(t) / \gamma_f - \dot{\theta}_g(t) / \gamma_g.
\end{align*}
\]
Noting Assumption 1, the above result leads to
\[
\begin{align*}
V \leq -e^T(t) Q e(t) / 2 + e^T(t) P b w(t) / 2 - g_0(e^T P b)^2 / r \\
&\quad + e^T(t) \dot{\theta}_t(t)(\xi(t) + \dot{\theta}_t(t) u(t)) - \dot{\theta}_g(t) / \gamma_f - \dot{\theta}_g(t) / \gamma_g.
\end{align*}
\] (22)
From the result in [1, Sec. 4.6.1], \(\text{proj}()\) ensures \(\dot{\theta}_t(t) \in \mathcal{D}_t\) and \(\dot{\theta}_g(t) \in \mathcal{D}_g\) with \(t \geq 0\), \(\forall \dot{\theta}_t(0) \in \mathcal{D}_t\) and \(\forall \dot{\theta}_g(0) \in \mathcal{D}_g\), and
\[
\begin{align*}
\dot{\theta}_t(t)(\xi(t) + \dot{\theta}_t(t) u(t)) - \dot{\theta}_g(t) / \gamma_f - \dot{\theta}_g(t) / \gamma_g 
\leq 0.
\end{align*}
\]
Applying the above inequalities to (22) yields
\[
V \leq -e^T(t) Q e(t) / 2 - g_0(e^T P b)^2 / r + e^T(t) P b w(t).
\]
By using \(r = 2\rho^2\), the above expression becomes
\[
V \leq -e^T(t) Q e(t) / 2 - g_0(e^T P b)^2 / (2\rho^2) - e^T(t) P b w(t) / (2\rho).
\]
Applying the triangle inequality to the above result and noting \(w = \omega_{e0} / \sqrt{2}\), one immediately gets
\[
V(t) \leq -e^T(t) Q e(t) / 2 + \rho^2 \omega_{e0}^2 / (2\rho^2).
\] (23)
Second, it follows from (17), (19) and (23) that
\[
\begin{align*}
V(t) &\leq -\lambda_2 V(t) + \lambda_3 (\gamma_f + \gamma_g) + (\rho \omega_{e0}^2 / (2\rho^2)) \\
&= -\lambda_4 (V(t) - \theta - c_g - (\rho \omega_{e0}^2 / (2\rho^2)))
\end{align*}
\]
with \(\lambda_4 = \min (Q)\) and \(\lambda_3 (P) \in \mathbb{R}^+\). Thus, one has
\[
V(t) \leq 0, \forall V(t) \geq c_f
\]
(24)
with \(c_f = c_f + c_g + (\rho \omega_{e0}^2 / (2\rho^2)) \in \mathbb{R}^+\). It follows from (x(0) \in \mathcal{D}_{e0} and \(\mathcal{D}_{e0} = \mathcal{D}_e \times \mathcal{D}_{e0}\) that \(e(0) \in \mathcal{D}_{e0}\). Using \(e(0) \in \mathcal{D}_{e0}\), the definition of \(\Omega_{e0}\) (12), (19) and (20), one obtains
\[
V(0) \leq c_f + c_g \leq c_f.
\] (25)
Using (24) and (25), one gets that the set
\[
\Omega_{e0} = \{ e(t) P e / 2 \leq c_{e0} \}
\]
is positively invariant, \(\forall c_e \leq c_{e0}\). Since \(w_0\) in (16) is independent of \(u_0\), one has that \(w_0\) in (17) is independent of \(\rho\). Thus, it follows from the definition of \(c_e\), \(\gamma_f\) and \(\gamma_g\) can be diminished by the decrease of \(\rho\). \(\gamma_f\) and/or \(\gamma_g\). From \(t = 2\rho^2\) and (19), increasing \(1 / r, \gamma_f\) and \(\gamma_g\) decreases \(\gamma_f\) and \(\gamma_g\), respectively. Thus, for any given designed and fixed \(c_e\) in (20), there exist suitably large \(1 / r, \gamma_f\) and \(\gamma_g\) to satisfy
\[
c_e \geq c_f + c_g + (\rho \omega_{e0}^2 / (2\rho^2))
\] (26)
resulting in \(c_e \geq c_{e0}\). Therefore, one obtains \(V(t) \leq c_f, t \geq 0\), which implies that the closed-loop system is stable in the sense of \(e(t) \in \mathcal{D}_e, \xi(t) \in \mathcal{D}_e, \) and \(u(t) \in \mathcal{D}_u\) with \(\mathcal{D}_{e0} = \{ e(t) P e / 2 \leq c_{e0} \}
\]
Integrating both sides of (23) at \(t = 0, T\) yields
\[
V(T) - V(0) \leq \frac{1}{2} \int_0^T e^T(t) Q e(t) dt + \frac{1}{2} \rho^2 \int_0^T w^2(t) dt
\]
with \(T \in [0, \infty)\). After simple manipulation on the above inequality, one gets the \(H^\infty\) tracking result (21).

If the selection of \(Q\) is restricted to \(\lambda_4 (Q) > \rho^2 (1 + \eta^2) / g_0\) (28) then one immediately obtains
\[
V(t) \leq -\lambda_4 (Q) \| e(t) \|^2 / 2 + \varphi
\]
with \(\lambda_4 = \lambda_4 (Q) - \rho^2 (1 + \eta^2) / g_0 \in \mathbb{R}^+\) and \(\varphi = (\rho \omega_{e0}^2 / (2\rho^2)) \in \mathbb{R}^+\). Thus, it is straightforward to get
\[
V(t) \leq -\lambda_4 (Q) \| e(t) \|^2 / 2 + \varphi
\] (29)
where \(\lambda_4 = \min (Q) \in \mathbb{R}^+\). Solving the above inequality using (1,
Lemma A.3.2], one obtains

\[ V(t) \leq V(0) \exp(-\lambda t) + (\xi^T + \xi_T + \rho^T \lambda_b). \]

The above result implies that \( e(t) \) converges to a set

\[ D_{\text{eq}} = \{ e(t) \leq \xi^T + \xi_T + \rho^T \lambda_b \} \]

that can be contracted to a small neighborhood of zero by the increase of \( 1/\rho, \lambda_{\text{min}}(Q), \gamma_f \) and \( \gamma_g \).

Remark 4. The results in Theorem 1 depend on the constraints in (20) and (26) related to two constants \( \gamma_f \) and \( \gamma_g \) in (19). Since the fuzzy approximation in (7) does not cover the compact sets \( \Omega_q \) and \( \Omega_g \) to which the optimal parameters \( \theta_q^* \) and \( \theta_g^* \) belong themselves depend on the compact set \( \Omega_q \). Hence, \( \Omega_q \) must be defined prior, and consequently, is independent of \( \Omega_d \) and \( \Omega_f \), which results in the requirement of large learning rates \( \gamma_f \) and \( \gamma_g \) to satisfy the inequalities (20) and (26) [42]. Noting the above control information, let \( a_n = [\theta^T, \phi^T, \rho^T] \) which is too small. Secondly, the adaptation also completely avoids the aforementioned two drawbacks. Finally, to determine the adaptive law (18), set \( \hat{\theta}_q(0) = 0 \) and \( \hat{\theta}_g(0) = [1.2, 1.2, \ldots, 1.2]^T \). Let \( \rho = 1/4, q = 10 \) and \( \gamma_f = 5 \) be initial values of the control parameters for all simulation cases. Simulations are carried out on MATLAB software running in Windows 7, where the step size is 0.001 s, and the other settings are kept default. To show the merits of the proposed approach, the indirect AFHC law (8) with (11) given by (21) is selected as a baseline controller.

A comparison of tracking performances between the two controllers under various values of \( \rho \) is shown in Fig. 1, where an integral absolute error (IAE) with respect to \( e \) and a logarithmic transformation \( N = \log_2(1/\rho) - 1 \) is applied to the vertical and horizontal axes, respectively for clear illustration. It is observed that these two approaches have very similar tracking performances. Nevertheless, the control law (8) has two drawbacks that can only be observed during simulation processes rather than from simulation results as follows.

4. An illustrative example

To verify the theoretical result in Theorem 1, consider an inverted pendulum that can be modeled by (1) with [37]

\[
\begin{align*}
\dot{x} & = g \sin x_1 - m_p \lambda_p x_2^2 \cos x_1 \sin x_1/(m_c + m_p) \\
\dot{x} & = 4l_p/3 - l_p m_p \cos^2 x_1/(m_c + m_p) \\
\dot{x} & = \cos x_1/(m_c + m_p) \\
\dot{x} & = 4l_p/3 - (l_p m_p \cos^2 x_1)/(m_c + m_p)
\end{align*}
\]

in which \( x_1 \) (rad) and \( x_2 \) (rad/s) are the angular position and velocity of the pendulum, respectively, \( g \) (m/s^2) is the gravitational acceleration, \( m_c \) (kg) is the mass of the cart, \( m_p \) (kg) is the mass of the pendulum, and \( l_p \) (m) is the half-length of the pendulum. For simulations, let \( m_c = 1, m_p = 0.1, g = 9.8, l_p = 0.5, d(t) = 5 \sin(t), x(0) = [\pi/6, 0]^T \) and \( \gamma_0(t) = (\pi/6)\sin(t) \). Noting the above control information, let \( \Omega_q = [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \) and \( \Omega_f = [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \times [-\pi/6, \pi/6] \). Therefore, one gets that \( \theta_q \) and \( \theta_g \) are finite. Consequently, the results in Theorem 1 depend on the constraints in (21) which is independent of \( \Omega_f \) and \( \Omega_g \), respectively. If a persistent-exciting condition is satisfied such that \( \theta_q \to 0 \) and \( \theta_g \to 0 \), the requirement on large \( \gamma_f \) and \( \gamma_g \) can be relaxed to a certain extent.

Remark 5. Compared with previous AFHC laws, the major differences of the proposed control law (14) with (18) are that the PD control term \( u_i \) in (14) is independent of \( \hat{g} \) and the adaptive laws in (18) are independent of \( u_i \). Compared with the previous H^\infty tracking result (13), the significance of the deduced H^\infty tracking result (21) is that the term \( w \) in (21) is independent of \( u_i \). The proof of Theorem 1 has successfully shown that the slight modification in the proposed control law (14) leads to several salient features as follows:

- Standard H^\infty robustness can be guaranteed because decreasing \( \rho \) in \( u_i \) enhances the H^\infty tracking result (21) without enlarging \( w = w_0/\hat{g}_0 \) where \( w_0 \) is given by (16);
- Since \( w_0 \) given by (17) is independent of \( \rho \), closed-loop stability can be guaranteed under a suitable choice of the approximation domain \( \Omega_q \), restricted by (20) and the control parameters \( \rho, Q, \gamma_f \) and \( \gamma_g \) restricted by (26) and (28).

Consequently, the proposed approach completely avoids the disadvantages of the classical indirect AFHC in [21] and some of its variations such as those in [27–30] without SMC compensation. The tracking performance can be enhanced by the increase of \( 1/\rho, \lambda_{\text{min}}(Q) \), \( \gamma_f \) and \( \gamma_g \), and the terms \( w_0 \) and \( g_0 \) in (26) do not need to be known in practice as the constraint (26) can be easily satisfied by trial-and-error. The cost of applying the proposed approach is that the global stability obtained by the indirect AFHC with SMC compensation is degraded to be local.

A comparison of tracking performances between the two controllers under various values of \( \rho \) is shown in Fig. 1, where an integral absolute error (IAE) with respect to \( e \) and a logarithmic transformation \( N = \log_2(1/\rho) - 1 \) is applied to the vertical and horizontal axes, respectively for clear illustration. It is observed that these two approaches have very similar tracking performances. Nevertheless, the control law (8) has two drawbacks that can only be observed during simulation processes rather than from simulation results as follows.

Firstly, since \( 1/\hat{g} \) (\( \theta_q^* \)) can be regarded as a variable gain of the PD control term \( u_i \) in (8), the performance of \( u_i \) in (8) can be affected by the settings of both the initial adaptive parameter \( \hat{\theta}_q(0) \) and the design parameters of the adaptive laws in (11), which is undesirable since the dependence of \( u_i \) on \( \hat{g} \) (\( \theta_q^* \)) makes \( H^\infty \) tracking blurry. Specifically, the tracking performance would be degraded while \( \hat{\theta}_q(0) \) and/or \( \hat{\theta}_g(0) \) are too small but the learning rate \( \gamma_g \) is too small. Secondly, the adaptation of \( \hat{\theta}_q \) in (11) can be diverged easily during simulations while the change rate of \( u_i \) (related to \( u_i \) and \( \rho \)) and \( \gamma_f \) are too large. A smaller step size can be adopted to solve this problem at the cost of increasing computational burden. Thus, these two drawbacks are detrimental in practice. The merits of the applied control law (14) are that it not only makes stable \( H_m \) tracking without SMC compensation possible, but also completely avoids the aforementioned two drawbacks.

A comparison of tracking performances by the proposed approach with various values of \( \rho, Q \) and \( \gamma_f \) is given in Fig. 2. Where logarithmic transformations \( N = \log_2(1/\rho) - 1, N = \log_2(q/10) + 1 \) and \( N = \log_2(\gamma_f) \) are applied to \( \rho, q \) and \( \gamma_f \), respectively to uniform horizontal axes. It is observed during simulations that the step size must be decreased to ensure simulation stability. Therefore, \( \rho, Q \) and/or \( \gamma_g \) are increased, which results in increased computational burden. It is worth noting that the closed-
loop system is unstable as long as \( q < 1 \) and/or \( \gamma_f < 1 \), which implies (26) and (28) are unsatisfied. Finally, Figs. 3–5 are also provided to further show how the control parameters \( \rho, Q \) and \( \gamma_f \) affect tracking accuracy in the proposed approach.

5. Conclusions

This study has successfully proven that an indirect AFHC approach without SMC compensation can guarantee stable \( H^\infty \) tracking under given initial conditions and parameter constraints. A slight modification on the certainty equivalent control law is significant since it not only makes stable \( H^\infty \) tracking without SMC compensation possible, but also alleviates some drawbacks of existing AFHC approaches for practical applications. Simulation results have verified correctness of the theoretical result. As the proposed method is based on the general class of affine nonlinear systems (1), the dissipativity and \( L_1 - L_\infty \) techniques in [43–45] can be naturally incorporated into this method to enhance its robustness against perturbations. This work would be done in further studies.

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References

[14] E.M.H. Balagli, R. Vatankhah, M. Broushaki, A. Alasty, Adaptive optimal multi-


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